

Common Limits of Fibonacci Circle Maps

Genadi Levin

Einstein Institute of Mathematics

Hebrew University

Givat Ram 91904, Jerusalem, ISRAEL

levin@math.huji.ac.il

Grzegorz Świątek

Department. of Mathematics and Information Science

Politechnika Warszawska

Plac Politechniki 1

00-661 Warszawa, POLAND

g.swiatek@mini.pw.edu.pl

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Abstract

We show that limits for the critical exponent tending to ∞ exist in both critical circle homeomorphism of golden mean rotation number and Fibonacci circle coverings. Moreover, they are the same. The limit map is not analytic at the critical point, which is flat, but has non-trivial complex dynamics.

1 Introduction

1.1 Limits of renormalization schemes.

Since the seminal work of Feigenbaum several renormalization schemes have been discovered in the dynamics of maps in dimension 1. They all share some common features: first return of a mapping from some topological class to suitably chosen intervals that shrink to the critical point (or value) of the dynamics when rescaled tend to a limit. This limit depends on the topological class of the original mapping and the exponent of the critical point, but is generally universal. Another key

feature is that the limit is a solution to some functional equation which gives its rescaling to a smaller interval in term of the fixed point and its rescalings to a larger scale. Concrete examples other than the Feigenbaum class include the critical circle homeomorphisms, Fibonacci unimodal maps and Fibonacci circle coverings.

Later, the dependence of these fixed points on the critical exponent was also examined in the situation when the exponent tends to ∞ , or sometimes to 1. So far, it has been done for the Feigenbaum class [8], [12] and golden mean critical circle mappings [3]. It turned out the limits exist, or at least there is a strong evidence for their existence, and they were of the same topological type and satisfied the same functional equation as the fixed point maps for finite values of the exponent.

In this paper we study Fibonacci circle coverings and homeomorphisms. They have been known to have fixed points of renormalization for odd integer values of the critical exponent, which belonged to certain distinct topological classes. For circle homeomorphisms the class that was generally used consisted of commuting pairs of maps on the interval. For coverings, the limits are box mappings with two branches. We have found that in both classes the fixed points tend to some limit class when the critical exponent grows to ∞ . What is surprising is that it is the same limit class in both cases. Topologically, the limit map can be viewed as a circle homeomorphism with the golden mean rotation number. Thus, when one starts in the class of critical circle covers, in the limit of infinite criticality there is a change of topological type: for example the degree of the map drops from 2 to 1.

Since this change was unexpected, one is left to wonder what may have caused it and when a similar phenomenon can be expected in the future. We can phrase our expectation in the form of a conjecture.

Conjecture 1 *Suppose that in a topological class of one-dimensional maps, each with one critical value, limits of appropriate renormalization schemes exist and satisfy a functional fixed-point equation, for each critical exponent from a sequence tending to ∞ . Then there exists a topologically different class of one-dimensional maps, for which limits of similar renormalization schemes exist and satisfy the same functional equation. Furthermore, in the limit as the exponent tends to ∞ in both classes, the fixed point maps tend to a common limit dynamics.*

The evidence in favor of this conjecture has so far been rather scant, so perhaps it should rather be viewed as an open problem which in our opinion is worth further study. The functional equation is the same for critical circle homeomorphisms and Fibonacci covers where we have found a common universal class. It is different for Fibonacci unimodal maps, and indeed although the limits for criticality tending to ∞ have not been studied in detail for this class, one can already say that they will have two branches and thus be different from any other known limits class.

A good test case for this conjecture would be to find a renormalization scheme leading to fixed points which satisfy the Feigenbaum equation, but in a topological class different from infinitely renormalizable unimodal maps. Another argument in support of the Conjecture follows from a discussion of the associate dynamics given in section 1.3.

1.2 Statement of the results.

Brief summary of the work. The methods used here and the general flavor of the results are the same as in an earlier work concerning the limits with infinite criticality for Feigenbaum maps. We build complex continuations of the dynamics and show that as the critical exponent tends to ∞ they lead to limits in a certain limit class of maps with flat critical points. This already turns out to be the same class for critical circle homeomorphisms and covers. In spite of not being analytic in a neighborhood of the critical point, this limit class shown to have a non-trivial complex dynamics. Out of this complex dynamics one can build McMullen towers and show their rigidity using the usual ideas. The rigidity of the towers implies the rigidity of the limits class itself which is shown to consist of just one mapping after a normalization. Out of this we derive the main theorem of the paper (Theorem 1) which does not mention directly the fixed points of renormalization, but is applicable directly to the underlying dynamics.

The classes of dynamics.

Definition 1.1 *Consider open intervals I^0, I^{-1}, I in the following configuration: $\bar{I}^0 \cap \bar{I}^{-1} = \emptyset$, $\bar{I}^0 \cup \bar{I}^{-1} \subset I$, $0 \in I^0$. The branches ψ^0, ψ^{-1} are defined on the corresponding intervals, are both monotone increasing, C^3 , and map onto I . Furthermore, each branch ψ^i , $i = 0, -1$, has exactly one critical point with the local representation as $(\xi^i(x))^\ell$ where ξ^i are diffeomorphisms and ℓ an odd integer bigger than 1. This implies that both branches have the same critical value at 0. The set of such mappings will be called \mathcal{G}_ℓ .*

Maps from \mathcal{G}_ℓ can be obtained naturally by inducing from critical circle coverings, see [14]. However, to use the results of that, and most other papers, one should consider branches $g^i(x) = \sqrt[\ell]{\psi^i(x^\ell)}$. The effect of this change of normalization is that there is only one critical point at $x = 0$.

Definition 1.2 *Consider open intervals I^0 and I^{-1} which share a common endpoint and I which is their convex hull. Suppose that $0 \in I^0$. There are branches ψ^0, ψ^{-1} defined from the respective intervals into I , both increasing, and after identifying the endpoints of I , the union of branches ψ^0 and ψ^{-1} extends to a circle homeomorphism.*

Each branch ψ^i is a diffeomorphisms of class C^3 and has a C^3 extension to a neighborhood of the common endpoint a^i of I and I^i in the form $\psi^i(x) = (\xi^i(x))^\ell$ where ξ^i is a local diffeomorphism and $\xi^i(a^i) = 0$. The class of such mappings with be denoted with \mathcal{G}_ℓ^1 .

Fibonacci combinatorics. Our use of the term Fibonacci combinatorics is somewhat different for maps of degree 1 and higher. For homeomorphisms, we simply mean that the rotation number is the golden mean.

A map g from \mathcal{G}_ℓ is said to have the Fibonacci combinatorics if there exists a weakly order preserving map from the circle to I with endpoints identified, which conjugates g on the forward orbit of the critical point to the dynamics of an orbit under the golden mean rotation.

Renormalization. Suppose that ψ is either a circle homeomorphism or a map from \mathcal{G}_ℓ with Fibonacci combinatorics. We normalize ψ so that the critical value is at 0. Then consider the sequence x_{q_n} where x_{q_n} is the preimage of 0 of order q_n , closest to 0 and q_n is the n -th Fibonacci number. Let I_n denote the interval between x_{q_n} and $x_{q_{n-1}}$. Let ψ_n denote the first return map of ψ into I_n . In particular, $\psi_n(0) = \psi^{q_n}(0)$.

Let ζ_n be a linear map specified by the condition $\zeta_n(\psi^{q_n}(0)) = 1$. Then, the n -th renormalization,

$$\mathcal{R}^n(\psi) = \zeta_n \circ \psi_n \circ \zeta_n^{-1}$$

is a map defined from a dense and open subset of $\zeta_n(I_n)$ into $\zeta_n(I_n)$.

From Theorem 1 in [14] we get the following:

Fact 1.1 *For each ℓ which is an odd integer greater than 1, there exists exactly one map $H_\ell \in \mathcal{G}_\ell$ with the Fibonacci combinatorics and constant $\tau_\ell < -1$ so that if $\phi = \phi^0, \phi^{-1}$ are the branches of H_ℓ , then*

- $\phi(0) = 1,$
-

$$\phi^{-1}(x) = \phi_{-1}(x)$$

for $x \in I^{-1}$ where the subscript k denotes a rescaling by τ_ℓ^{-k} , e.g., $\phi_{-1}(x) = \tau \circ \phi \circ \tau^{-1}$,

- the fixed point equation holds for all $x \in \tau_\ell^{-1}I^0$:

$$\phi_1(x) = \phi_{-1} \circ \phi(x) . \tag{1}$$

- for any $\psi \in \mathcal{G}_\ell$ with the Fibonacci combinatorics, the sequence of renormalizations $\mathcal{R}^n(\psi)$ converges to H_ℓ uniformly on the domain of H_ℓ .

Similarly, from [4] and [5] we get this:

Fact 1.2 *For each ℓ which is an odd integer greater than 1, there exists exactly one map $H_\ell^1 \in \mathcal{G}_\ell^1$ with the Fibonacci combinatorics and constant $\tilde{\tau}_\ell < -1$ so that if $\phi = \phi^0, \phi^{-1}$ are the branches of H_ℓ^1 , then*

- $\phi(0) = 1,$
-

$$\phi^{-1}(x) = \phi_{-1}(x)$$

for $x \in I^{-1}$ where the subscript k denotes a rescaling by $\tilde{\tau}_\ell^{-k}$,

- *the fixed point equation (1) holds for all $x \in \tilde{\tau}_\ell^{-1}I^0$,*
- *for any $\psi \in \mathcal{G}_\ell^1$ with the Fibonacci combinatorics, the sequence of renormalizations $\mathcal{R}^n(\psi)$ converges to H_ℓ^1 uniformly on the domain of H_ℓ^1 .*

The main theorem and its corollaries.

Theorem 1 *There exist $x_0 < 0$ and $\tau < -1$ for which the following holds. Consider a sequence of all odd integers ℓ . For each ℓ consider a map ψ_ℓ which is either in \mathcal{G}_ℓ or \mathcal{G}_ℓ^1 with the Fibonacci combinatorics. Let $\mathcal{R}^n(\psi_\ell)$ be the sequence of renormalizations. Then, there exists a sequence k_ℓ such that for any sequence $n_\ell \geq k_\ell$ for all ℓ , mappings $\mathcal{R}^{n_\ell}(\psi_\ell)$ converge almost uniformly on the set $(x_0, x_0/\tau) \cup (x_0/\tau, x_0\tau)$ as $\ell \rightarrow \infty$. This limit is independent of the sequence ψ_ℓ and is a homeomorphism of the circle obtained by identifying x_0 with τx_0 with the golden mean rotation number. It further belongs the \mathcal{EWF} -class defined later, see Definition 1.4.*

Corollary 1.1 *The scaling factors τ_ℓ and $\hat{\tau}_\ell$ introduced in Facts 1.1 and 1.2 tend to a common limit $\tau < -1$.*

The limit $\tilde{\tau}_\ell \rightarrow \tau$ as $\ell \rightarrow \infty$ was the subject of an experimental study, see [3]. In particular, numerically, $\tau = -3.71\dots$

The next result does not follow formally from the main theorem, but can be derived from its proof.

Theorem 2 *The Hausdorff dimension of the post-critical set for maps in \mathcal{G}_ℓ with the Fibonacci combinatorics, which depends only on ℓ by [14], tends to 1 as ℓ tends to ∞ .*

Note that for Feigenbaum maps of the interval, the Hausdorff dimension of the attractor tends to a limits which is less than 1, see [11]. It is related to the fact that in the Feigenbaum case the topological dynamics does not change in the limit as $\ell \rightarrow \infty$, but in the case of Fibonacci covers studied in the present paper, such a

change occurs. Further comments on this phenomenon follow in connection with the associated dynamics of G .

In the complex plane, this difference disappears and the Hausdorff dimensions of the Julia sets for Feigenbaum maps were shown to tend to 2 in [13]. For Fibonacci covers there is no proof in the literature, but we expect the arguments of [13] to work with minor changes.

1.3 The limit class \mathcal{EWF} .

Let us first recall the concept of Poincaré neighborhoods:

Definition 1.3 *If I is an open interval, then $\mathcal{D}(I)$ denotes the geometric disk centered at the midpoint of I . Similarly, for $0 < \alpha \leq \pi$, we write $\mathcal{D}(I, \alpha)$ to denote the set of point z in the plane such that the circle passing through z and the endpoints of I intersects the real line at angle less than α (small α meaning a small section of the disk).*

In particular, $\mathcal{D}(I, \pi)$ is the doubly slit plane $(\mathbb{C} \setminus \mathbb{R}) \cup I$.

A star added to the notation of a disk, or disk neighborhood defined above, will mean a punctured neighborhood with the point 0 removed. For example, $\mathcal{D}^*(-1, 2)$ is equivalent to $\mathcal{D}(-1, 2) \setminus \{0\}$.

Definition 1.4 *We start by specifying the map on the real line.*

Fix parameters $x_0 < 0$, $\tau < -1$ and $\tau^2 > R > \tau x_0$, assuming also $\frac{x_0}{\tau} < 1 < \tau x_0$. We consider a mapping ϕ continuous on an interval $[x_0, R']$, $R' > x_0/\tau$, and a real-analytic orientation-preserving diffeomorphism from (x_0, R') onto its image $(0, R)$. Suppose that $\phi(x_0) = 0$, $\phi(0) = 1$, $\phi(x_0/\tau) = \tau x_0$ and $\phi(x_0/\tau^2) = x_0/\tau$.

We can also consider mappings $\phi_k = \tau^{-k}\phi\tau^k$ for $k \in \mathbb{Z}$. One can easily see that $\phi_{-1}(x_0\tau) = 0$ and $\phi_{-1}(x_0/\tau) = \tau\phi(x_0/\tau^2) = x_0$. After identifying points x_0 and τx_0 and putting ϕ on $[x_0, x_0/\tau]$ and ϕ_{-1} on $[x_0/\tau, x_0\tau]$, we get a degree 1 circle homeomorphism \mathcal{H} .

We assume that the rotation number of \mathcal{H} is the golden mean, $\frac{1}{1+\frac{1}{1+\dots}}$ and that functional equation (1) holds for $x \in [x_0/\tau^2, x_0/\tau]$.

Furthermore, ϕ has an analytic continuation, also denoted by ϕ , and we make the following assumptions about it:

1. ϕ is defined on a topological disk U and $U \cap \mathbb{R} = (x_0, R')$. Also, U is symmetric with respect to the real axis.
2. ϕ is a covering (unbranched) of the punctured disk $V := D(0, R) \setminus \{0\}$ by U .
3. For any $0 < r \leq R$, $\phi^{-1}(D(0, r) \setminus \{0\})$ is contained in $\mathcal{D}(x_0, r')$ where r' is real and $\phi(r') = r$. It implies that $U \subset \mathcal{D}(x_0, R')$.

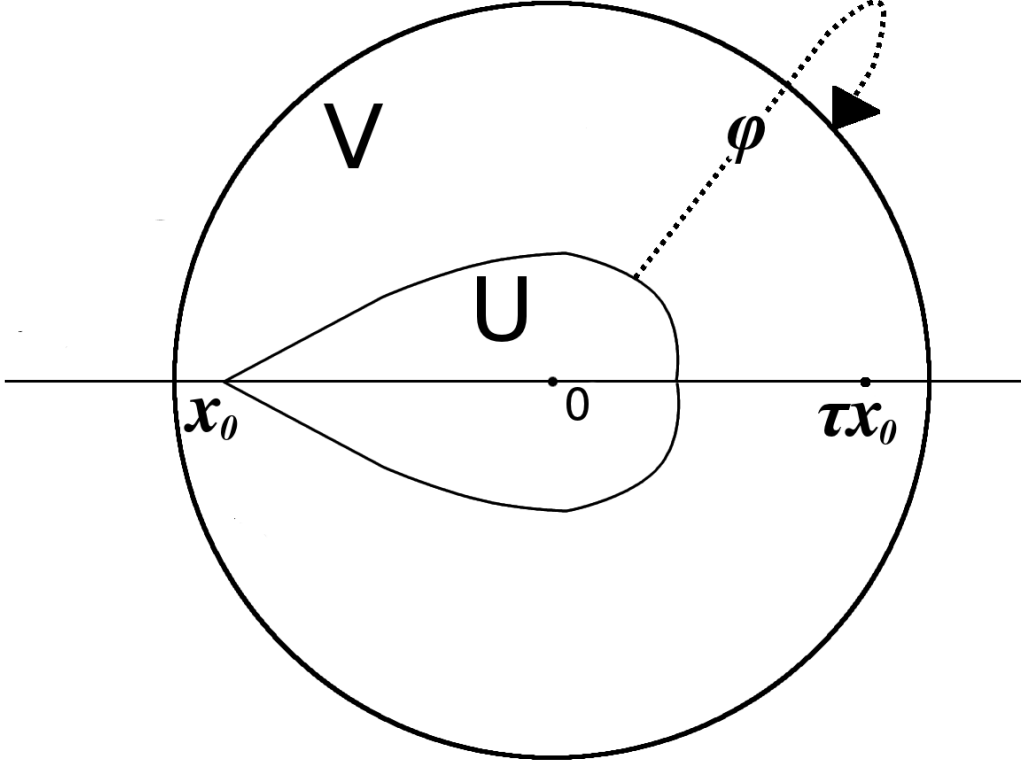


Figure 1: The mapping ϕ .

4. If I is a real segment which does not contain 0 and ϕ^{-1} denotes the inverse branch of ϕ which maps I into \mathbb{R} , then for any $0 < \alpha \leq \pi$ $\phi^{-1}(\mathcal{D}(I, \alpha)) \subset \mathcal{D}(\phi^{-1}(I), \alpha)$.
5. Define the mapping $G(x) := \tau^{-1}\phi\tau^{-1}$. By previous hypotheses, it fixes x_0 and is analytic in its neighborhood. Assume that G has the following power series expansion at x_0 :

$$G(x) = x - \epsilon(x - x_0)^3 + O(|x - x_0|^4)$$

with $\epsilon > 0$.

The class of all mappings ϕ with these properties will be denoted with \mathcal{EWF} .

Theorem 3 *Class \mathcal{EWF} consists of one mapping.*

Associated dynamics of G . An insight into the nature of the flat critical point of ϕ , but also the bifurcation which occurs for the limit dynamics and allows one to unfold it as either a circle homeomorphism, or a covering map, can be gained from looking at the associated dynamics of the function G . Function G appears in item 5. of Definition 1.4. Its dynamical interpretation comes from the functional equation (1) and is stated as Lemma 1.1, or simply:

$$\phi \circ G = \tau^{-2} \phi . \quad (2)$$

Since x_0 is a neutral but topologically attracting fixed point of G , equation (2) implies that it has to be a flat critical point of ϕ . Further information about this point can be gained from the interpretation of $\log \phi$ as a Fatou coordinate for this point. Perhaps more interestingly, one can consider the bifurcation of this fixed point which corresponds to considering the dynamics of fixed points of renormalization H_ℓ or H_ℓ^1 for ℓ large but finite. There are two ways of bifurcating a neutral, topologically attracting fixed point for a real analytic map with negative Schwarzian derivative. One will create a pair of attracting fixed points and a repelling one between them, all on the real line. The other is to make the fixed point on the real line attracting, and form a pair of repelling ones in the complex plane.

The second bifurcation regime is easier to understand since it leads to no topological change of the dynamics on the real line. The first mode creates a repelling fixed point on the real line. By equation (7), a repelling fixed point of G_ℓ corresponds to a singularity where H_ℓ goes to ∞ , see also Lemma 2.1. Hence, the dynamics between the attracting points has unbounded image for every ℓ even though its domain shrinks to a point as ℓ goes to ∞ . The dynamics of H_ℓ outside the interval between the attracting fixed points undergoes no bifurcation at the limit. However, the dynamics between the fixed points vanishes in the limit and this is the reason why the topological degree drops.

As an example of the consequences of this, we can recall the difference in limiting behavior of the Hausdorff dimension of the attractors for Feigenbaum polynomials and circle covers. For Feigenbaum polynomials the second type bifurcation occurs which leads to no change of dynamics of the real line as a consequence no qualitative change of the Hausdorff dimension. As will be show in this paper, for circle coverings the first type of bifurcation happens which leads to the disappearance of a part of dynamics and closing of gaps on Cantor sets in the limit.

Finally, the big difference between two types of bifurcation is only relevant on the real line. So, one does not expect to see it when studying the Hausdorff dimension of Julia sets, for example.

1.4 Basic properties of \mathcal{EWF} maps.

Connection between ϕ and the Fatou coordinate. Observe that $\log \phi$ is a well defined univalent map. Its inverse can be defined as the lifting of \exp to the universal covering ϕ .

Lemma 1.1 *The transformation $h(z) = \frac{\log \phi}{\log \tau^{-2}}$ is a Fatou coordinate for G :*

$$h \circ G(z) = h(z) + 1$$

for all $z \in U$.

This provides useful information about h , and therefore the singularity of ϕ at x_0 , in the light of uniqueness of the Fatou coordinate. It leads to the next geometrical lemma.

Lemma 1.2 *For any $\epsilon > 0$ there is $\delta > 0$ and for each $z \in U$, if $|z - x_0| < \delta$, then $|\arg(z - x_0)| < \frac{\pi}{4} + \epsilon$.*

Proof. The key to the proof of this Lemma is Lemma 1.1 and the transformation h introduced there. By its definition, the range of h is contained in a certain right half-plane $\Re w > A_1$. Next, we consider a standard construction of the Fatou coordinate following for example [1], which gives $\tilde{h} = \xi(\frac{C}{(z-x_0)^2})$ where $\xi(z) = z + O(|z|^{-1/2})$ and $C > 0$. Since a relevant inverse branch of $\frac{C}{(z-x_0)^2}$ maps the set $\{w : \Re w > 0\}$ inside an angle $|\arg(z - x_0)| < \frac{\pi}{4}$, the preimage $\tilde{h}^{-1}(\{w : \Re w > A_2\})$ is also contained in the same angle if A_2 is chosen sufficiently large.

By the uniqueness of the Fatou coordinate $h(z) - \tilde{h}(z) = T$ and so $W := h^{-1}(\{w : \Re w > A_2 + T\})$ is contained in the same angle. Finally, choose k so that $k > A_2 + T - A_1$. Then, by Lemma 1.1, the domain U is contained in $G^{-k}(W)$. Since G is conformal, the claim follows. □

Based on the interpretation of $\log \phi$ as a Fatou coordinate up to a normalization, see [1], one gets:

Fact 1.3 *$\log \phi$ extends to a quasiconformal mapping of the Riemann sphere sending x_0 to ∞ .*

Analytic continuation of the functional equation.

Lemma 1.3 *Suppose that ϕ a mapping from the \mathcal{EWF} -class. Then the functional equation (1) holds for every argument $z \in \tau^{-1}U$, meaning also that both sides of the functional equation are well defined.*

Proof. We will show a topological disk W such that $W \cap \mathbb{R} = (a, x_0/\tau)$, $\phi_{-1} \circ \phi(a) = R/\tau$, and $\phi_{-1} \circ \phi$ is defined on W and a covering of $D(0, R/\tau) \setminus \{0\}$. By item 3 of Definition 1.4, ϕ_{-1} provides a universal covering of $D(0, R/\tau) \setminus \{0\}$ by some topological disk W_1 such that $W_1 \cap \mathbb{R} = (a_1, \tau x_0)$, $W_1 \subset \mathcal{D}(a_1, \tau x_0)$ and

$$a_1 = \phi_{-1}^{-1}(R/\tau) > \phi_{-1}^{-1}\tau = \tau\phi^{-1}(1) = 0.$$

Then $W = \phi^{-1}(W_1)$ where ϕ^{-1} denotes a univalent inverse branch which maps the real trace $(a_1, \tau x_0)$ to its preimage $(a, x_0/\tau)$.

Then looking at ϕ_1 on its domain $\tau^{-1}U$, we observe that it is a universal covering of $D(0, R/\tau) \setminus \{0\}$. So ϕ_1 and $\phi_{-1} \circ \phi$ are universal coverings of the same set and are equal on a segment $(x_0/\tau^2, x_0/\tau)$ by equation (1). Then the lifting of the identity is a univalent map from W onto U which is the identity on the segment and thus globally.

□

2 Critical circle covers

This section is devoted to the proof of the following theorem.

Theorem 4 *Take a sequence of odd integers ℓ_n tending to ∞ . Consider a sequence of fixed point maps H_{ℓ_n} and scaling constants τ_{ℓ_n} introduced in Fact 1.1 and let ϕ_{ℓ_n} denote their branches whose domains contain 0. Let x_n denote the critical point of the ϕ_{ℓ_n} . For a subsequence n_k the following are true:*

- *sequences x_{n_k} and $\tau_{\ell_{n_k}}$ converge to x_0 and τ , respectively, which satisfy the inequalities postulated by Definition 1.4,*
- *mappings $\phi_{\ell_{n_k}}$ converge almost uniformly on $(x_0, x_0/\tau)$ to ϕ which belongs to the \mathcal{EWF} -class.*

2.1 Fixed-point equations

Map near the critical point Fix an odd integer $\ell \geq 3$, and let H_ℓ be the map from the Fact 1.1. Consider also the corresponding map h_ℓ near the critical point. In other words, if $p(x) = x^\ell$ is the change of variable on \mathbb{R} , then $h_\ell = p^{-1} \circ H_\ell \circ p$. It consists of two branches $g_i = p^{-1} \circ \phi_i \circ p$, where ϕ_i , $i = 0, -1$ are the branches of H_ℓ , so that g_i is defined on $J^i = p^{-1}(I^i)$ with the common image $J = p^{-1}(I)$. The scaling factor for h_ℓ is $\alpha_\ell = \tau_\ell^{1/\ell} < -1$. The first return map of h_ℓ to the central interval J^0 retained to those components in J^0 , which intersect the forward critical orbit $\{h^i(0)\}_{i \geq 0}$ consists of the central branch which is $g_{-1} \circ g_0$ and is defined on $\alpha^{-1}J^0$, and the off-central branch which is g_0 and is defined on $\alpha_\ell^{-1}J^1$. Furthermore, after the rescaling by $x \mapsto \alpha_\ell x$, this first return

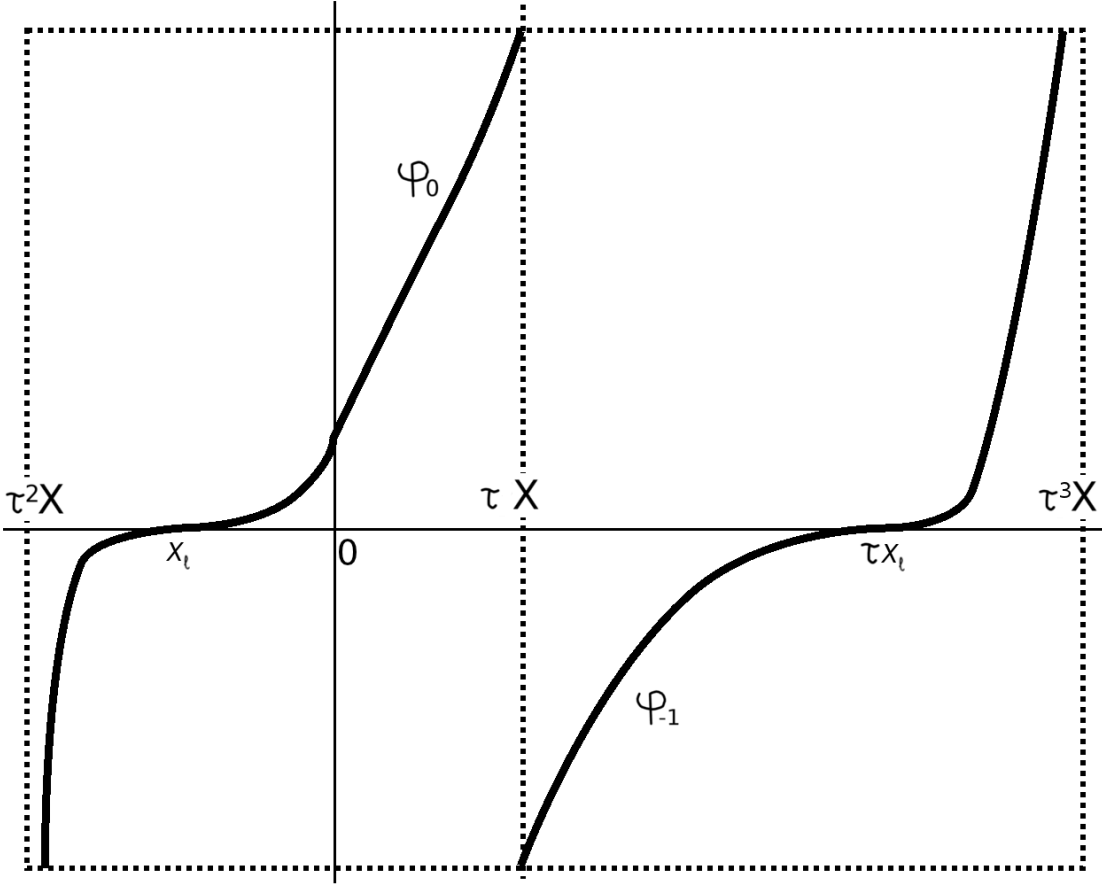


Figure 2: The graph of H_ℓ .

map coincides with the initial map h_ℓ . In other words, $\alpha_\ell^{-1}J^1 \subset J^0$ and $\alpha_\ell J^0 = J$, and g_0, g_{-1} satisfy the functional equations

$$g_{-1} = \alpha \circ g_0 \circ \alpha^{-1}, \quad (3)$$

$$g_0 = \alpha \circ g_{-1} \circ g_0 \circ \alpha^{-1}. \quad (4)$$

Since $g_0(x) = E(x^\ell)$, where a diffeomorphism $E = E_\ell$ belongs to the Epstein class, it follows that the map h_ℓ has a non-positive Schwarzian derivative. Recall that a real-analytic diffeomorphism belongs to the Epstein class provided that its inverse has an analytic continuation to the upper half-plane. Note that E maps I^0 onto J .

For every $n \geq 1$, consider the first return map h^n of h_ℓ to the interval $J_n = \alpha_\ell^{-n}J$ (so that $h^1 = h_\ell$ and $J_1 = J^0$) retained to the components in J_n intersecting the forward critical orbit. As it follows from (3), the map h^n is equal to $\alpha_\ell^{-n} \circ h_\ell \circ \alpha_\ell^n$. h^n consists of a pair of homeomorphisms $g_n = \alpha_\ell^{-n} \circ g_0 \circ \alpha_\ell^n : J_{n+1} \rightarrow J_n$ and $g_{n-1} = \alpha_\ell^{-n+1} \circ g_0 \circ \alpha_\ell^{n-1} : \alpha^{-n}J^1 \rightarrow J^n$. By the Fibonacci combinatorics, the first entry of the h_ℓ -iterates of 0 to the interval J_n occurs at the time q_n , where $q_0 = 1, q_1 = 2, q_2 = q_0 + q_1 = 3, \dots$ are the Fibonacci numbers.

Analytic continuation Fix ℓ . In this paragraph, we sometimes drop the index ℓ denoting $\tau = \tau_\ell, \alpha = \alpha_\ell$ etc. Let $\phi = \phi_0$, the branch of H which contains 0 in its domain I^0 . Recall that ϕ satisfies the equation

$$\tau^{-2}\phi(x) = \phi \circ \tau^{-1} \circ \phi \circ \tau^{-1}(x), \quad x \in I_c^0. \quad (5)$$

We define

$$G_\ell = \tau_\ell^{-1} \circ \phi_\ell \circ \tau_\ell^{-1}. \quad (6)$$

Then (5) turns into

$$\tau_\ell^{-2}\phi_\ell = \phi_\ell \circ G_\ell. \quad (7)$$

Denote by y_ℓ the unique zero of $g_0 : J^0 \rightarrow J$. Set $x_\ell = p(x_\ell) = y_\ell^\ell$. Then x_ℓ is the only zero and the only critical point of ϕ on I^0 . Denote $I = (A, B)$. Then $I^0 = (B/\tau, A/\tau)$ and $B/\tau < x_\ell < 0 < A/\tau$.

Lemma 2.1 (a) $\phi : (B/\tau, A/\tau) \rightarrow (A, B)$ has a real analytic continuation to $\phi : (A, B) \rightarrow \tau^2(A, B)$.

(b) The restriction $\phi : (x_\ell, \tau x_\ell) \rightarrow (0, \tau^2)$ is a diffeomorphism, which belongs to the Epstein class.

(c) $\phi(1/\tau) = 1/\tau^2, \phi'(1/\tau) = 1, \phi(x_\ell/\tau) = \tau x_\ell$. Also, $1/|\tau| < |A| < |\tau|, 1 < \tau x_\ell < B < |\tau|^2$.

(d) (associated dynamics of G). There exists a unique point $X \in (1/\tau, 0)$, such that $\phi(X) = \tau X$. Then $\tau\phi(x_\ell/\tau^2) < \tau^2 X < x_\ell$, and these 3 points are the only

real fixed points of G . Moreover, the points $\tau\phi(x_\ell/\tau^2)$ and x_ℓ are attracting with the multiplier $1/\alpha^2$ while the point τ^2X is strictly repelling.

(e) mapping ϕ extends to a real-analytic homeomorphism from $(\tau^2X, \tau x_\ell)$ onto $(-\infty, \tau^2)$, such that $\phi = E^\ell$, where $E : (\tau^2X, \tau x_\ell) \rightarrow (-\infty, \alpha^2)$ is a diffeomorphism from the Epstein class.

Proof. Map G_ℓ , here abbreviated to G , is given by formula (6). It is real-analytic on (A, B) and $G((A, B)) = (B/\tau, A/\tau)$. Then (a) follows from (5). Since $\phi = (E)^\ell$ and E is in the Epstein class, then the inverse map $\phi^{-1} : (0, B) \rightarrow (x_\ell, A/\tau)$ extends to a univalent map $\phi^{-1} : \mathbb{C}_{(0, B)} \rightarrow \mathbb{C}_{(x_\ell, A/\tau)}$, and hence, for a real branch of the map G^{-1} , there is a univalent extension $G^{-1} : \mathbb{C}_{(B/\tau, 0)} \rightarrow \mathbb{C}_{(A, \tau x_\ell)}$. Since $B/\tau < x_\ell$, it allows us to define a univalent map $\phi^{-1} = G^{-1} \circ \phi^{-1} \circ \tau^{-2}$ from $\mathbb{C}_{(0, \tau^2)}$ into $\mathbb{C}_{(x_\ell, \tau x_\ell)}$, where it is a diffeomorphism on the real line. Finally, let us show (c). We use (5). First, $\tau^{-2} = \tau^{-2}\phi(0) = \phi \circ \tau^{-1}\phi \circ \tau^{-1}(0) = \phi(1/\tau)$. Besides, $\tau^{-2}\phi'(0) = \tau^{-2}\phi'(1/\tau)\phi'(0)$, i.e., $\phi'(1/\tau) = 1$. Also, $\phi(G(x_\ell)) = \tau^{-2}\phi(x_\ell) = 0$, which implies that $G(x_\ell) = \tau^{-1}\phi(x_\ell/\tau) = x_\ell$. To prove the rest of (c), notice that, by the combinatorics, $B > \tau x_\ell > \phi(0) = 1 > A/\tau$. Coupled with the inequality $1/|\tau| < |A/B| < |\tau|$, this implies the inequalities of (c).

(d) The function $\tau^{-1}\phi$ is strictly decreasing in a neighborhood of $[1/\tau, 0]$, and $\tau^{-1}\phi(1/\tau) = 1/\tau^3 > 1/\tau$, $\tau^{-1}\phi(0) = 1/\tau < 0$. Hence, it has a unique fixed point $X \in (1/\tau, 0)$. Introduce $\Gamma = \tau \circ \phi \circ \tau^{-2}$. By the functional equation, $\Gamma^2 = G$. But $\Gamma(\tau^2X) = \tau^2X$. Therefore, $\Gamma(x_\ell), \tau^2X, x_\ell$ are fixed points of G . Let us show that $\tau\phi(x_\ell/\tau^2) < \tau^2X < x_\ell$. Indeed, as for the map h , the points y_ℓ/α_ℓ^2 and y_ℓ/α_ℓ lie in the central interval I^0 . Hence $y_\ell/\alpha_\ell < g_0(y_\ell/\alpha_\ell^2) \in I^1$. Then $x_\ell/\tau < \phi(x_\ell/\tau^2)$, i.e. $x_\ell < \Gamma(x_\ell)$. In turn, since Γ is strictly decreasing, this implies that $\Gamma(x_\ell) < \tau^2X < x_\ell$. Furthermore, since $\phi(x_\ell + x) = ax^\ell + o(|x|^\ell)$, expanding G into the power series and substituting in (5) yields $G'(x_\ell) = G'(\Gamma(x_\ell)) = \alpha^{-2}$. That is, $x_\ell, \Gamma(x_\ell)$ are attracting fixed points of G . On the other hand, G is in the Epstein class, in particular, $SG \leq 0$. It follows that τ^2X is the only other fixed point of G , and it is strictly repelling.

(e) For any $x \in (\tau^2X, x_\ell)$, we find $n \geq 0$ such that $G^n(x) \in (B/\tau, x_\ell)$. Then we can define $\phi(x) = \tau^{2n}\phi(G^n(x))$. Clearly, $n \rightarrow \infty$ as $x \rightarrow \tau^2X$, i.e. $\phi(x) \rightarrow -\infty$ as $x \rightarrow \tau^2X$. It follows that E in the decomposition $\phi = E^\ell$ extends to a real-analytic diffeomorphism $E : (X, \tau x_\ell) \rightarrow (-\infty, \alpha^2)$. It remains to check that this extension is still in the Epstein class. Indeed, we have an univalent map $E^{-1} : \mathbb{C}_{(A^{1/\ell}, B^{1/\ell})} \rightarrow \mathbb{C}_{(B/\tau, A/\tau)}$. By the above, $E^{-1} : (A^{1/\ell}, B^{1/\ell}) \rightarrow (B/\tau, A/\tau)$ extends to a real-analytic diffeomorphism from $(-\infty, \alpha^2)$ onto $(X, \tau x_\ell)$. By the Uniqueness Theorem for analytic functions, we are done.

□

2.2 Bounds for covering maps

Real bounds.

Proposition 1 *There exist two constants $1 < T_1 < T_2 < \infty$, such that $T_1 < |\tau_\ell| < T_2$, for all H_ℓ .*

The proof is contained in the following two lemmas 2.2, 2.3.

Lemma 2.2 *There exists $1 < T_1$, such that $T_1 < |\tau_\ell|$ for all H_ℓ .*

Proof. Fix ℓ large. It is enough to prove that there is $C > 0$ independent on ℓ such that $|\alpha_\ell| \geq 1 + C/\ell$. We will drop the index ℓ in $\alpha_\ell, \tau_\ell, h_\ell$ etc.

Consider the first entry map h_{n-1} to the interval J_{n-1} , for n large enough. Let us apply the shortest interval argument to the collection of pairwise disjoint intervals $J_n^j = h^j(J_n)$, $j = 0, 1, \dots, q_{n-1} - 1$. Denote by J_n^k the shortest interval of this collection. Consider two cases.

1. $k = 0$, i.e., J_n is the shortest interval. The intervals $h^{q_{n-2}}(J_n)$ and $h^{q_{n-3}}(J_n)$ lie on the opposite sides from J_n and both are subsets of J_{n-3} . Therefore, by the assumption, $|J_{n-3}| \geq 3|J_n|$. On the other hand, $|J_{n-3}|/|J_n| = |\alpha|^3$, hence, $|\alpha| \geq 3^{1/3}$.

2. $k \geq 1$. Denote $J = (a, b)$. As α is negative and $J_1 = \alpha^{-1}J \subset J$, then $|\alpha|^{-1} \leq |a|/b \leq |\alpha|$. Since J_n is the interval between the points $a/\alpha^n, b/\alpha^n$, the origin divides the interval J_n into two intervals with the ratio of their lengths at most $|\alpha|$. Now, we can assume that $|\alpha| < 1 + 1/\ell$ because otherwise there is nothing to prove. Therefore, the above ratio is at most $1 + 1/\ell$. Consider $J_n^1 = h(J_n) = E(p(J_n))$, where, as before, $p(x) = x^\ell$, and $E = E_\ell$ is a diffeomorphism near 0, $E(0) = 1$. Hence, given ℓ , $|E'(x)/E'(y)| < 2$ for $x, y \in J_n$ provided n is large. We get the following *fact*: there is $C_0 > 1$ independent on ℓ , such that, provided n is large enough, the point 1 divides the interval J_n^1 into two intervals with the ratio of their lengths at most C_0 . Denote by $J_n^{j_1}, J_n^{j_2}$ two neighbors of J_n^k from the right and from the left (if there is no neighbor from one side, add an interval on this side of the length $|J_n^k|$). Denote by K the smallest interval containing $J_n^{j_1}, J_n^{j_2}$. Let us pull back K by a branch of h^{k-1} from J_n^k to J_n^1 . Along the way, a preimage of either $J_n^{j_1}$ or $J_n^{j_2}$ can turn into I_n^1 at most once. If it happens, i.e., say, for $i = j_1 - 1$, $h^{-i}(J_n^{j_1}) = J_n^1$, the point 1 splits $h^{-i}(K)$ into two parts. Then we cut off the part of $h^{-i}(K)$ which does not contain $h^{-i}(J_n^k)$, and continue to pull back. Since h has a non-positive Schwarzian and by the above *fact*, there exists some $C_1 > 0$ independent on ℓ , such that, for every n large enough, a C_1 -neighborhood $K_1 = \{x : \text{dist}(x, J_n^1) < C_1|J_n^1|\}$ of the interval J_n^1 contains at most one interval on each side of J_n^1 from the collection of intervals J_n^j , $j = 0, 1, \dots, q_{n-1} - 1$. As J_{n-5} contains at least two intervals from our collection on each side of J_n , K_1 is contained in $h(J_{n-5})$. Now we use that n is large and get

$3C_1 \leq |h(J_{n-5})|/|h(J_n)| \leq 2|\tau|^5$, i.e., $|\tau|^5 \geq C_2$, where $C_2 = 3C_1/2$ is independent on ℓ .

□

Lemma 2.3 *There exists T_2 such that for all H_ℓ , we get $|\tau_\ell| < T_2$.*

Proof. Consider the map $\phi_{-1} = \tau \circ \phi \circ \tau^{-1}$. By Lemma 2.1, $\phi_{-1} : (\tau^2 x_\ell, \tau x_\ell) \rightarrow (\tau^3, 0)$ is a diffeomorphism from the Epstein class. On the other hand, $\phi_{-1} = (E_{-1})^\ell$, where $E_{-1} = \alpha \circ E \circ \tau^{-1}$. Hence, $E_{-1} : (\tau^2 x_\ell, \tau x_\ell) \rightarrow (\alpha^3, 0)$ is a diffeomorphism, from the Epstein class, too.

Let us estimate from above the $|E'_{-1}(1)|$. Consider the infinitesimal cross-ratio formed by points $\tau^2 x_\ell, 0, 1, 1+dx$. Note that $E_{-1}(\tau^2) = \alpha^3$, $E_{-1}(0) = \alpha$ and, from Lemma 2.1(c), $E_{-1}(1) = 1/\alpha$. Since E_{-1} is in the Epstein class, the cross-ratio inequality gives

$$|E'_{-1}(1)| \frac{|\alpha^3 - \alpha|}{|\tau^2 x_\ell|} \frac{1}{|1/\alpha - \alpha|} \frac{|1 - \tau^2 x_\ell|}{|1/\alpha - \alpha^3|} \leq 1$$

Note that $\frac{|1 - \tau^2 x_\ell|}{|\tau^2 x_\ell|} > 1$. Let us calculate $E'_{-1}(1)$. We use that $\phi'_{-1} = \ell E_{-1}^{\ell-1} E'_{-1}$ and $E_{-1}(1)^{\ell-1} = (1/\alpha)^{\ell-1} = |\alpha|/|\tau|$. On the other hand, $\phi = \tau \circ \phi_{-1} \circ \phi \circ \tau^{-1}$ and so $\phi'(0) = \phi'_{-1}(1)\phi'(0)$. Thus $\phi'_{-1}(1) = 1$. We get

$$1 \leq \ell \frac{|\alpha|}{|\tau|} \frac{|1/\alpha - \alpha||1/\alpha - \alpha^3|}{|\alpha^3 - \alpha|} = \frac{|1 - |\tau|^{4/\ell}|}{|\tau|^{1+2/\ell}}. \quad (8)$$

We can use the inequality $1 - x < \log(1/x)$, which holds for $0 < x < 1$, to obtain from (8)

$$|\tau| \leq \ell |\tau|^{2/\ell} \log(|\tau|^{4/\ell}).$$

Now it is easy to conclude that there is T_2 independent on $\ell \geq 3$, such that $|\tau| < T_2$.

□

2.3 Limit maps

We follow general lines of [12] although some modifications and changes are necessary. Our aim is to pick a convergent subsequence from ϕ_{ℓ_m} by some kind of compactness argument. As $\ell_m \rightarrow \infty$, then the domains of definition tend to degenerate at a limit of the critical points x_{ℓ_m} . To deal with this phenomenon, we consider inverse branches of ϕ_{ℓ_m} corresponding to values to the right of the point x_{ℓ_m} .

By Lemma 2.1, each ϕ_ℓ can be represented as $(E_\ell(x))^\ell$ with E_ℓ an Epstein diffeomorphism with the range at least onto the interval $(\sqrt[\ell]{A_\ell}, \sqrt[\ell]{\tau_\ell^2})$. Further from the same Lemma, one gets that $E_\ell^{-1}(\mathcal{D}(\sqrt[\ell]{A_\ell}, \sqrt[\ell]{\tau_\ell^2})) \subset \mathcal{D}(B_\ell/\tau_\ell, \tau_\ell x_\ell)$. In the light of Proposition 1 and Lemma 2.1 (c), by taking a subsequence we may assume that $\tau_{\ell_m} \rightarrow \tau > 1$ and $A_{\ell_m} \rightarrow A$, $B_{\ell_m} \rightarrow B$. Choosing yet another subsequence, we may assume that $x_{\ell_m} \rightarrow x_0$. Note that $1/|\tau| \leq |A| \leq |\tau|$, $1 \leq \tau x_0 \leq B \leq |\tau|^2$.

We will actually invert not ϕ_{ℓ_m} , but its lifting to the universal cover, which is defined as follows. Consider the universal cover of the punctured disk $\mathcal{D}^*(\sqrt[\ell]{A_\ell}, \sqrt[\ell]{\tau_\ell^2})$ by \exp and apply to the cover the linear map $w \mapsto \ell_m w$. Then we get a $2\pi\ell_m$ -periodic domain Π_m , which contains the left half-plane and is bounded by a curve $x = \gamma_m(y)$ ($w = x + iy$), where $\gamma_m(0) = \log \tau_{\ell_m}^2$ and $\gamma_m(\pm\pi\ell_m) = \log A_{\ell_m}$. The real branch of the lifting of ϕ_{ℓ_m} , which maps onto a right neighborhood of x_{ℓ_m} , has a complex extension

$$P_{\ell_m}(w) := E_{\ell_m}^{-1}(\exp(w/\ell_m)) \quad (9)$$

which is defined in Π_m and maps it into $\mathcal{D}(B_\ell/\tau_\ell, \tau_\ell x_\ell)$ and so P_{ℓ_m} are uniformly bounded. By Montel's theorem we can pick a subsequence m_k , such that $P_{\ell_{m_k}}$ converges to a mapping P . Its domain of definition is $\Pi_\infty := \{w : \Re w < \log \tau^2\}$. Indeed, since $\exp(w/\ell_m)$ maps an open arc of γ_m between the points with $y = \pm\pi\ell_m$ bijectively onto the boundary of $\mathcal{D}(\sqrt[\ell_m]{A_{\ell_m}}, \sqrt[\ell_m]{\tau_{\ell_m}^2})$ (strictly, speaking, without the real point $\sqrt[\ell]{A_\ell}$), and since $A_{\ell_m}, \tau_{\ell_m}$ converge, then it is easy to see that every compact subset of Π_∞ belongs to Π_m for almost all m , and vice versa. Since the domains vary with m , they should be normalized for example by precomposing with a translation, which tends to 0 in the limit. This implies uniform convergence $P_{\ell_{m_k}} \rightarrow P$ on compact subsets.

Let us see that P is non-constant. Note that $0 \in \Pi_\infty$. As $P_{\ell_m}(0) = 0$, then $P(0) = 0$. Besides, points $\log(1/\tau_{\ell_m}^2)$ converge to the point $\log(1/\tau^2)$, which lies in the left half-plane, i.e., in Π_∞ . Hence, using Lemma 2.1(c), $P(\log(1/\tau^2)) = 1/\tau$. In particular, P is not a constant function.

It is also clear that P is univalent. This is because for any compact subset of Π_∞ and ℓ_m large enough, P_{ℓ_m} is univalent on this set, which is evident from their defining formulas (9).

Let us define $x_0^- := \lim_{x \rightarrow -\infty} P(x)$. Since $(P)^{-1}$ is increasing to the right of x_0^- and $x_{\ell_m} < P_{\ell_m}(x)$ for every m and real negative x , we must have $x_0 \leq x_0^-$. Let us also note that $x_0^- < P(\log(1/\tau^2)) = 1/\tau$.

We will next show that, for any $b < \tau^2$, the image of the half-plane $\{w : \Re w < \log b\}$ by the limit map P is contained in $\mathcal{D}((x_0, b'), \pi/2)$ where $b' = P(\log b)$.

The inclusion will follow once we show that, for any $w = x + iy$ with $x < \log b$

and for m large enough (depending on w),

$$\exp(w/\ell_m) \in \mathcal{D}((0, \sqrt[\ell_m]{b}), \pi/2). \quad (10)$$

The inclusion then follows from the definition of P and since E_{ℓ_m} is in the Epstein class. In turn, (10) can be checked directly by showing that

$$\frac{|\exp(\frac{w}{\ell}) - \frac{1}{2} \exp(\frac{\log b}{\ell})|}{\frac{1}{2} \exp(\frac{\log b}{\ell})} = |2 \exp \frac{w - \log b}{\ell} - 1| < 1, \quad (11)$$

if $\Re w < \log b$ and ℓ is large enough.

We can now define a limit mapping ϕ which will be shown to belong to the \mathcal{EWF} class.

Fix any $\tau x_0 < R < \tau^2$ and define $\Pi_* = \{w : \Re w < \log R\}$. Consider P on Π_* . We set $U = P(\Pi_*)$. Then $\phi|_U := \exp \circ (P)^{-1}$. The intersection of U and the real axis is an interval (x_0^-, R') , where $R' = P(\log R)$.

We have shown that ϕ_{ℓ_m} converge to ϕ uniformly on any compact subset of $(x_0^-, R']$.

As $[\log(1/\tau^2), 0] \subset \Pi_*$ and $P(0) = 0$, $P(\log(1/\tau^2)) = 1/\tau$, there is a complex neighborhood S_1 of the interval $[1/\tau, 0]$, such that $\phi_{\ell_m} \rightarrow \phi$ uniformly in S_1 . In particular, $x_0 \leq x_0^- < 1/\tau < 0$ and ϕ is analytic in S_1 .

Recall that $G_{\ell_m}(z) = \tau_m^{-1} \phi_{\ell_m}(\tau_m^{-1} z)$. We define $G(z) = \lim_{m \rightarrow \infty} G_{\ell_m}(z)$ wherever the limit exists. G is defined and analytic in a complex neighborhood S_2 of 0, and $\phi = \tau^2 \circ \phi \circ \tau^{-1} \phi \circ \tau^{-1}$ in S_2 . Indeed, for a small enough disk S_2 centered at 0, $\tau^{-1} S_2 \subset S_2 \subset S_1$, so that $G = \tau^{-1} \circ \phi \circ \tau^{-1}$ is well-defined and analytic in S_2 , and $G(S_2)$ is a small neighborhood of $G(0) = 1/\tau$. As $1/\tau \in S_1$, we may pass to the limit uniformly in S_2 in the equations $\phi_{\ell_m} = \tau_{\ell_m}^2 \circ \phi_{\ell_m} \circ G_{\ell_m}$.

Now we can extend ϕ to an analytic map which is defined in a complex neighborhood S_3 of the interval $[0, 1]$ as follows. Since $\tau^{-1}[0, 1] = [1/\tau, 0] \subset S_1$, then $G = \lim G_{\ell_m}$ is also defined and analytic in a complex neighborhood S_3 of $[0, 1]$, and $G(S_3)$ is a neighborhood of $\tau^{-1} \phi([1/\tau, 0]) = [1/\tau, 1/\tau^3]$, where the last interval is contained in S_1 . Then we define in S_3 : $\phi = \tau^2 \circ \phi \circ G$. Since $S_2 \subset S_3$, then we get an analytic continuation of ϕ . It is also clear that $\phi_{\ell_m} \rightarrow \phi$ uniformly in a neighborhood of $[1/\tau, 1]$, and ϕ is strictly increasing in $[x_0^-, 1]$.

As the next step, since $\tau^{-1}[x_0, 0] \subset [0, 1]$, $G = \lim G_{\ell_m}$, uniformly in a neighborhood S_4 of $[x_0, 0]$. In particular, $G(x_0) = x_0$, and x_0 is topologically non-repelling: $|G(x) - x_0| \leq |x - x_0|$ for every $x \in [x_0, 0]$.

Lemma 2.4 $G([x_0, x_0^-]) = [x_0, x_0^-]$.

Indeed, otherwise $G(x_0^-) < x_0^-$, and there is $x' \in (x_0^-, 0)$ such that $G(x') = x_0^-$. Then $\phi(x') = \tau^2 \circ \phi \circ G(x') = 0$, where $x' > x_0^-$, a contradiction. This proves the lemma.

G_{ℓ_m} converge to G uniformly on a complex neighborhood of $[x_0, 0]$. We use now Lemma 2.1 (d). Since $G'_{\ell_m}(x_{\ell_m}) = \sqrt[\ell_m]{\tau_m^{-2}}$, the convergence implies $G'(x_0) = 1$. Coupled with the information that x_0 is topologically non-repelling on both sides, this implies the power-series expansion:

$$G(z) - x_0 = (z - x_0) + a(z - x_0)^{q+1} + O(|z - x_0|^{q+1})$$

with some $a \leq 0$ and some q even. First, we prove that $a \neq 0$, i.e. G is not the identity. If $G(z) = z$, then, for every x near 0, $\phi(x) = \phi(G(x)) = \phi(x)/\tau^2$, i.e. $\phi(x) = 0$, a contradiction. Thus, $a < 0$.

Now we prove that $q = 2$ considering a perturbation. There is a fixed complex neighborhood W of x_0 , such that the sequence of maps $(G_{\ell_m})^{-1}$ are well-defined in W and converges uniformly in W to G^{-1} . Since each ϕ_{ℓ_m} belongs to the Epstein class, then each $(G_{\ell_m})^{-1}$ extends to a univalent map of the upper (and lower) half-plane into itself. It extends also continuously on the real line, and has there exactly 3 fixed point $\Gamma_{\ell_m}(x_{\ell_m})$, $\tau_m^2 X_{\ell_m}$, x_{ℓ_m} , where $\tau_m^2 X_{\ell_m}$ is strictly repelling. Therefore, by the Wolff-Denjoy theorem, every point in either half-plane is attracted to $\tau_m^2 X_{\ell_m}$ by the iterates of $(G_{\ell_m})^{-1}$, in particular, G_{ℓ_m} has no non-real fixed points. This implies $q = 2$ by Rouché's principle.

Finally, we show that $x_0 = x_0^-$. Indeed, otherwise x_0^- would be a fixed point of G to the right of x_0 . Then, for big m , G_{ℓ_m} would have either a fixed point in the upper half-plane or a fourth real fixed point, a contradiction.

3 Critical circle homeomorphisms

The goal of this Section is to prove the following theorem.

Theorem 5 *Take a sequence of odd integers ℓ_n tending to ∞ . Consider a sequence of fixed point maps $H_{\ell_n}^1$ and scaling constants $\tilde{\tau}_{\ell_n}$ introduced in Fact 1.2 and let ϕ_{ℓ_n} denote their branches whose domains contain 0. Let x_n denote the critical point of the ϕ_{ℓ_n} . For a subsequence n_k the following are true:*

- *sequences x_{n_k} and $\tilde{\tau}_{\ell_{n_k}}$ converge to x_0 and τ , respectively, which satisfy the inequalities postulated by Definition 1.4,*
- *mappings $\phi_{\ell_{n_k}}$ converge almost uniformly on $(x_0, x_0/\tau)$ to ϕ which belongs to the \mathcal{EWF} -class.*

3.1 Fixed-point equations

Map near critical point Fix an odd integer $\ell \geq 3$, and let H_ℓ^1 be the map from the Fact 1.2. Consider the corresponding map \tilde{h}_ℓ near the critical point, i.e., $\tilde{h}_\ell = p^{-1} \circ H_\ell^1 \circ p$, where $p(x) = x^\ell$. It consists of two branches $\tilde{g}_i = p^{-1} \circ \phi_i \circ p$, where

ϕ_i , $i = 0, -1$ are the branches of H_ℓ^1 , so that \tilde{g}_i is defined on $J^i = p^{-1}(I^i)$ with the common image $J = p^{-1}(I)$. The intervals J_0, J^{-1} have a common endpoint. The scaling factor for \tilde{h}_ℓ is $\tilde{\alpha}_\ell = \tilde{\tau}_\ell^{1/\ell} < -1$. The functional equations for H_ℓ^1 imply that \tilde{g}_i satisfy similar equations:

$$\tilde{g}_{-1} = \tilde{\alpha}_\ell \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell^{-1}(x), \quad (12)$$

$$\tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell = \tilde{g}_{-1} \circ \tilde{g}_0. \quad (13)$$

Furthermore, by [4], [5], \tilde{g}_0 extends in a real-analytic fashion through the left end point of J^0 to a neighborhood of the interval $[\tilde{\alpha}_\ell, 0]$, and similarly \tilde{g}_{-1} extends to a real-analytic homeomorphism defined in a neighborhood of the interval $[0, 1]$, so that

$$\tilde{g}_0 \circ \tilde{g}_{-1} = \tilde{g}_{-1} \circ \tilde{g}_0 \quad (14)$$

near the point 0. Finally, $\tilde{g}_0(x) = E(x^\ell)$ where a diffeomorphism $E = E_\ell$ from a neighborhood of $[\tilde{\alpha}_\ell, 0]$ onto its image belongs to the Epstein class.

Comment 1 *The pair of maps $f_- = \tilde{g}_0 : [1/\tilde{\alpha}_\ell, 0] \rightarrow [\tilde{g}_0(1/\tilde{\alpha}_\ell), 1]$, $f_+ = \tilde{\alpha}_\ell^{-1} \circ f_- \circ \tilde{\alpha}_\ell : [0, 1] \rightarrow [f_+(0), f_+(1)]$ is a commuting pair in the sense of [4], [5], which is the unique fixed point of the renormalization operator corresponding to the golden mean rotation number. To be more precise, the equations (12)- (13), (14) mean that*

$$\tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell = \tilde{g}_{-1} \circ \tilde{g}_0 = \tilde{g}_0 \circ \tilde{g}_{-1},$$

and they are equivalent to the following two conditions:

$$f_+ = \alpha \circ f_+ \circ f_- \circ \alpha^{-1}, \quad f_+ \circ f_- = f_- \circ f_+.$$

We introduce a point y_ℓ , which is the unique zero of \tilde{g}_0 in $[\alpha, 0]$. Define also an associated dynamics $\gamma = \tilde{\alpha}_\ell \circ g_0 \circ \tilde{\alpha}_\ell^{-2}$. Note that the map γ depends on ℓ .

Lemma 3.1 (a) \tilde{g}_0 extends to a real-analytic orientation preserving homeomorphism from $(\tilde{\alpha}_\ell^2 y_\ell, \tilde{\alpha}_\ell y_\ell)$ onto $(\tilde{\alpha}_\ell, \tilde{\alpha}_\ell^2)$, where it has a representation $\tilde{g}_0(x) = E(x^\ell)$ with E a diffeomorphism in the Epstein class. In particular, $\tilde{\alpha}_\ell^2 y_\ell < \tilde{\alpha}_\ell$.

(b) $\gamma : (\tilde{\alpha}_\ell^2 y_\ell, 0) \rightarrow (\tilde{\alpha}_\ell, 0)$ is an orientation reversing diffeomorphism, which has a unique fixed point y_ℓ . Moreover, $\gamma'(y_\ell) = 1/\tilde{\alpha}_\ell \in (-1, 0)$. We have: $\tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 = \tilde{g}_0 \circ \gamma$ wherever both sides are well-defined.

(c) $\gamma^2 = \tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell^{-1} : (\tilde{\alpha}_\ell^2 y_\ell, 0) \rightarrow (\tilde{\alpha}_\ell, 1/\tilde{\alpha}_\ell)$ is an orientation preserving diffeomorphism, which has a unique fixed point at y_ℓ .

Proof. We have formally: $\tilde{g}_0 \circ \gamma = \tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0$. On the other hand, both sides are well-defined near y_ℓ , and $\gamma(y_\ell) \in (\tilde{\alpha}_\ell, 0)$. Hence, $\tilde{g}_0(\gamma(y_\ell)) = 0$ implies that $\gamma(y_\ell) = y_\ell$. In particular, $\tilde{g}_0'(y_\ell)\gamma'(y_\ell) = \tilde{\alpha}_\ell^{-1}\tilde{g}_0'(y_\ell)$, hence, $\gamma'(y_\ell) = \tilde{\alpha}_\ell^{-1}$. We have: $\tilde{\alpha}_\ell \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 = \tilde{\alpha}_\ell^{-1} \circ \tilde{g}_0 \circ \tilde{\alpha}_\ell$. Applying this for $x = 0$ we get $\tilde{g}_0(1/\tilde{\alpha}_\ell) = 1/\tilde{\alpha}_\ell^2$.

Since $0 > y_\ell > \tilde{\alpha}_\ell$, we can write: $y_\ell/\tilde{\alpha}_\ell = \tilde{g}_0(y_\ell/\tilde{\alpha}_\ell^2) > \tilde{g}_0(1/\tilde{\alpha}_\ell) = 1/\tilde{\alpha}_\ell^2$, i.e., $\tilde{\alpha}_\ell y_\ell > 1$ and $\tilde{\alpha}_\ell^2 y_\ell < \tilde{\alpha}_\ell$. But γ is a diffeomorphism of a neighborhood of $[\tilde{\alpha}_\ell^2 y_\ell, 0)$ onto a neighborhood of $(\tilde{\alpha}_\ell, 0]$. Hence, the formula $\tilde{g}_0 = \tilde{\alpha}_\ell \circ \tilde{g}_0 \circ \gamma$ gives us an analytic continuation of \tilde{g}_0 to a neighborhood of $[\tilde{\alpha}_\ell^2 y_\ell, 0)$, with $\tilde{\alpha}_\ell^2 y_\ell$ the only critical point and with $\tilde{g}_0(\tilde{\alpha}_\ell^2 y_\ell) = \tilde{\alpha}_\ell$.

Formally, $\gamma^2 = \tilde{\alpha}_\ell^{-1} \circ \gamma \circ \tilde{\alpha}_\ell^{-1}$. The latter map is an orientation preserving diffeomorphism of a neighborhood of $(0, \tilde{\alpha}_\ell y_\ell]$ onto a neighborhood of $(1/\tilde{\alpha}_\ell, 0]$. Then the formula $\tilde{g}_0 = \tilde{\alpha}_\ell^2 \circ \tilde{g}_0 \circ \gamma^2$ defines an analytic continuation of \tilde{g}_0 to $(0, \tilde{\alpha}_\ell y_\ell]$ with $\tilde{\alpha}_\ell y_\ell$ the only critical point and with $\tilde{g}_0(\tilde{\alpha}_\ell y_\ell) = \tilde{\alpha}_\ell^2 \tilde{g}_0(0) = \tilde{\alpha}_\ell^2$. The rest follows easily. \square

Let us come back to the H_ℓ^1 . It consists of the pair $\phi_j = p \circ \tilde{g}_j \circ p^{-1}$, $j = 0, -1$, which are defined on the intervals $(x_\ell, x_\ell/\tilde{\tau}_\ell)$ and $(x_\ell/\tilde{\tau}_\ell, \tilde{\tau}_\ell x_\ell)$ respectively, where $x_\ell = y_\ell^\ell$. Notice that $\phi_0(x) = (E(x))^\ell$. Recall that

$$\phi_{-1} = \tau \circ \phi_0 \circ \tau^{-1}, \quad \phi_0 = \tau \circ \phi_{-1} \circ \phi_0 \circ \tau^{-1}. \quad (15)$$

We denote

$$\tilde{\Gamma}_\ell = p_\ell \circ \gamma \circ p_\ell^{-1}, \quad \tilde{G}_\ell = \tilde{\Gamma}_\ell^2.$$

Then Lemma 3.1 is reformulated as follows (except for the part (d) below, which still needs to be proved).

Lemma 3.2 (a) $\phi = \phi_0$ extends to a real-analytic orientation preserving homeomorphism from $(\tilde{\tau}_\ell^2 x_\ell, \tilde{\tau}_\ell x_\ell)$ onto $(\tilde{\tau}_\ell, \tilde{\tau}_\ell^2)$, where it has a representation $\phi(x) = (E(x))^\ell$ with $E = E_\ell$ a diffeomorphism in the Epstein class, $E(0) = 1$.

(b) $\tilde{\Gamma}_\ell : (\tau^2 x_\ell, 0) \rightarrow (\tau, 0)$ is an orientation reversing diffeomorphism, which has a unique fixed point at x_ℓ . Moreover, $\tilde{\Gamma}_\ell'(x_\ell) = 1/\tilde{\alpha}_\ell \in (-1, 0)$. We have: $\tilde{\tau}_\ell^{-1} \circ \phi = \phi \circ \tilde{\Gamma}_\ell$ wherever both sides are well-defined.

(c) $\tilde{G}_\ell = \tilde{\tau}_\ell^{-1} \circ \phi \circ \tilde{\tau}_\ell^{-1} : (\tilde{\tau}_\ell^2 x_\ell, 0) \rightarrow (\tilde{\tau}_\ell, 1/\tilde{\tau}_\ell)$ is an orientation preserving diffeomorphism, which has a unique fixed point at x_ℓ .

(d) $\phi(1/\tilde{\tau}_\ell) = 1/\tilde{\tau}_\ell^2$, $\phi'(1/\tilde{\tau}_\ell) = 1$, $\phi(x_\ell/\tilde{\tau}_\ell) = \tilde{\tau}_\ell x_\ell$. Also, $1 < \tilde{\tau}_\ell x_\ell < \tilde{\tau}_\ell^2$.

It remains to check (d), and it is done very similar to the proof of the part (c) of Lemma 2.1 as the equations are identical. First, $\tilde{\tau}_\ell^{-2} = \tilde{\tau}_\ell^{-2} \phi(0) = \phi \circ \tilde{\tau}_\ell^{-1} \phi \circ \tilde{\tau}_\ell^{-1}(0) = \phi(1/\tilde{\tau}_\ell)$. Besides, $\tilde{\tau}_\ell^{-2} \phi'(0) = \tilde{\tau}_\ell^{-2} \phi'(1/\tilde{\tau}_\ell) \phi'(0)$, i.e., $\phi'(1/\tilde{\tau}_\ell) = 1$. Finally, $\tilde{\tau}_\ell^2 = \phi(\tilde{\tau}_\ell x_\ell) > \tilde{\tau}_\ell x_\ell = \phi(x_\ell/\tilde{\tau}_\ell) > 1 = \phi(0)$.

3.2 Bounds

Real bounds.

Proposition 2 *There exist two constants $1 < T_1 < T_2 < \infty$, such that $T_1 < |\tilde{\tau}_\ell| < T_2$, for all H_ℓ^1 .*

The proof is contained in the following two lemmas 3.3, 3.4.

Lemma 3.3 *There exists $1 < T_1$, such that $T_1 < |\tilde{\tau}_\ell|$ for all H_ℓ^1 .*

Proof. We use an idea from [20] Let f be a critical circle homeomorphism with the golden mean rotation number, a single critical value at $0 = f(c)$, where c is the critical point of an integer odd order ℓ , and f is C^3 with negative Schwarzian. Then, by [4], [5] ϕ on $[1/\tilde{\tau}_\ell, 0]$ is the uniform limit of a sequence of maps $A_n \circ f^{q_n} \circ A_n^{-1}$, where q_n is the Fibonacci sequence and A_n is a linear map (from the angle coordinate of the unit circle into reals), which maps $f^{q_n}(0)$ to 1. We will consider also $f_t = f - t$, for small real $t > 0$, and denote, for N integer and $0 \leq t' \leq t$, $N(t') = f_{t'}^N(0)$. Note that when t moves back to 0, all positive iterates of any point x move to the right, and all negative move to the left. Let us fix a minimal positive t , such that f_t has a periodic orbit of period q_{n+2} . It implies the following fact (*), to which we will often refer:

(*) for every point x , the ordering on the circle of points $f_{t'}^m(x)$, for $0 \leq m \leq q_{m+2} - 1$, does not depend on $t' \in [0, t]$.

We denote by $|(a, b)|$ the length of an interval with the end points a and b .

Claim. *There is $K > 0$ such that, for every ℓ and every n large enough, either*

$$\frac{|(q_n(t), 2q_n(t))|}{|(0, q_n(t))|} \geq K, \quad (16)$$

or

$$\frac{|(-2q_n(t), -q_n(t))|}{|(-q_n(t), 0)|} \geq K, \quad (17)$$

Let us see the Claim implies the Lemma. Assume (16) holds, for a fixed ℓ and every n big enough. Note that, $0 < -q_{n+1}(0) < q_n(t) < 2q_n(t) < 2q_n(0) < -q_{n-3}(0)$. On the other hand, as $n \rightarrow \infty$, $-q_{n-3}(0)/-q_{n+1}(0) \rightarrow \tilde{\tau}_\ell^4$. Hence, for n big,

$$\tilde{\tau}_\ell^4 > \frac{|(-q_{n+1}(0), 2q_n(0))|}{|(0, -q_{n+1}(0))|} > \frac{|(q_n(t), 2q_n(t))|}{|(0, q_n(t))|} \geq K.$$

Similarly, let (17) hold. Then $0 > q_{n+1}(0) > -q_n(t) > -2q_n(t) > -2q_n(0) > q_{n-3}(0)$ while, as $n \rightarrow \infty$, $q_{n-3}(0)/q_{n+1}(0) \rightarrow \tilde{\tau}_\ell^4$. Hence, for n big,

$$\tilde{\tau}_\ell^4 > \frac{|(-2q_n(0), q_{n+1}(0))|}{|(q_{n+1}(0), 0)|} > \frac{|(-2q_n(t), -q_n(t))|}{|(-q_n(t), 0)|} \geq K.$$

Let us prove the Claim. Consider a partition of the unit circle by the points, $N(t)$, for $0 \leq N \leq q_{n+2} - 1$. Note that $q_{n+2}(t) = 0$. Denote by J the shortest interval of this partition.

(1) $J_0 = (0, q_n(t))$ is an interval of the partition. Indeed, if it contains some $i(t)$, $1 \leq i \leq q_{n+2} - 1$, then, by (*), the same holds with $t = 0$, a contradiction, because $q_{n+2}(0)$ is the first return of 0 to $(0, q_n(0))$.

Consider the configuration of 4 points $-q_n(t), 0, q_n(t), 2q_n(t)$. First, we apply to it $f_t^{q_{n+1}}$.

(2) Let us check that $F_t = f_t^{q_{n+1}}$ is a diffeomorphism on $(-q_n(t), 2q_n(t))$ except for the last iterate when $(q_n + q_{n+1} - 1)(t)$ is the critical point of f_t .

(i) $F_t(q_n(t), 2q_n(t)) = (0, q_n(t))$, hence, by (1), F_t is a diffeomorphism on $(q_n(t), 2q_n(t))$.

(ii) $F_t(J_0) = (q_{n+1}(t), 0)$. The latter interval contains no $i(t)$, $1 \leq i \leq q_{n+1} - 1$, because this is true for $t = 0$. Thus F_t is a diffeomorphism on J_0 .

(iii) $F_t = f_t^{q_{n+1}} \circ f_t^{q_n}$. Since $f_t^{q_n}((-q_n(t), 0)) = J_0$, $f_t^{q_n}$ is a diffeomorphism on $(-q_n(t), 0)$. Also, $f_t^{q_{n+1}}$ is a diffeomorphism on J_0 . Indeed, $f_t^{q_{n+1}}(J_0) = (q_{n+1}(t), q_{n+2}(t))$, and the latter interval contains no $i(t)$, $0 \leq i \leq q_{n+1} - 1$, because the same is true for $t = 0$ (we again use (*)).

Thus (2) checked. By this, if $f_t^i(J_0) = J$, for some $0 \leq i \leq q_{n+1} - 1$, then we employ the shortest interval argument and arrive at (16). If $f_t^{q_{n+1}}(J_0) = (-q_n(t), 0)$ is J , then we get immediately (17).

(3) Otherwise, we must meet J when applying f_t^i (for some $1 \leq i \leq q_n - 1$) to the interval $(-q_n(t), 0)$ inside of the configuration $-2q_n(t), -q_n(t), 0, q_n(t)$. We have checked in (2) that $f_t^{q_n}$ is a diffeomorphism on $[-q_n(t), q_n(t)]$. But $f_t^{q_n}$ is a diffeomorphism on $(-2q_n(t), -q_n(t))$, too, because otherwise $(-q_n(t), 0)$ contains some $i(t)$, $1 \leq i \leq q_n - 1$, hence, J_0 contains $(i + q_n)(t)$. Since $i + q_n < q_{n+2}$, this is impossible by the step (1). Thus $f_t^j((-q_n(t), 0)) = J$, for some $1 \leq j \leq q_n - 1$ where f_t^j is a diffeomorphism on $(-2q_n(t), q_n(t))$. Then, by the shortest interval argument, (17) follows.

□

Lemma 3.4 *There exists T_2 such that for all H_ℓ^1 , we get $|\tilde{\tau}_\ell| < T_2$.*

Proof. Consider the map $\phi_{-1} = \tau \circ \phi \circ \tau^{-1}$. By Lemma 3.2 (a),

$$\phi_{-1} : (\tilde{\tau}_\ell^2 x_\ell, \tilde{\tau}_\ell x_\ell) \rightarrow (\tilde{\tau}_\ell^3, 0)$$

is a diffeomorphism from the Epstein class. Then we can proceed word by word as in the proof of Lemma 2.3.

□

3.3 Limit maps

Consider inverse branches of ϕ_{ℓ_m} corresponding to values to the right of the point x_{ℓ_m} . By Lemma 3.2, each ϕ_ℓ can be represented as $(E_\ell(x))^\ell$ with $E_\ell :$

$(\tilde{\tau}_\ell^2 x_\ell, \tilde{\tau}_\ell x_\ell) \rightarrow (\sqrt[\ell]{\tilde{\tau}_\ell}, \sqrt[\ell]{\tilde{\tau}_\ell^2})$ an Epstein diffeomorphism. Hence,

$$E_\ell^{-1}((\mathcal{D}(\sqrt[\ell]{\tilde{\tau}_\ell}, \sqrt[\ell]{\tilde{\tau}_\ell^2})) \subset \mathcal{D}(\tilde{\tau}_\ell^2 x_\ell, \tilde{\tau}_\ell x_\ell).$$

In the light of Proposition 2 and Lemma 3.2, by taking a subsequence we may assume that $\tilde{\tau}_{\ell_m} \rightarrow \tau > 1$ and $x_{\ell_m} \rightarrow x_0$. Note that $1 \leq \tau x_0 \leq |\tau|^2$.

Consider the universal cover of the punctured disk $\mathcal{D}^*(\sqrt[\ell]{\tilde{\tau}_\ell}, \sqrt[\ell]{\tilde{\tau}_\ell^2})$ by \exp and apply to the cover the linear map $w \mapsto \ell_m w$. Then we get a $2\pi\ell_m$ -periodic domain Π_m , which contains the left half-plane and is bounded by a curve $x = \gamma_m(y)$ ($w = x + iy$), where $\gamma_m(0) = \log \tilde{\tau}_{\ell_m}^2$ and $\gamma_m(\pm\pi\ell_m) = \log \tilde{\tau}_{\ell_m}$. The real branch of the lifting of ϕ_{ℓ_m} , which maps onto a right neighborhood of x_{ℓ_m} , has a complex extension

$$P_{\ell_m}(w) := E_{\ell_m}^{-1}(\exp(w/\ell_m)) \quad (18)$$

which is defined in Π_m and maps it into $\mathcal{D}(\tilde{\tau}_\ell^2 x_\ell, \tilde{\tau}_\ell x_\ell)$ and so P_{ℓ_m} are uniformly bounded. Pick a subsequence m_k , such that $P_{\ell_{m_k}}$ converge to a mapping P . Its domain of definition is $\Pi_\infty := \{w : \Re w < \log \tau^2\}$. This implies uniform convergence $P_{\ell_{m_k}} \rightarrow P$ on compact subsets.

Let us see that P is non-constant. Note that $0 \in \Pi_\infty$. As $P_{\ell_m}(0) = 0$, then $P(0) = 0$. Besides, points $\log(1/\tilde{\tau}_{\ell_m}^2)$ converge to the point $\log(1/\tilde{\tau}^2)$, which lies in the left half-plane, i.e., in Π_∞ . Hence, using Lemma 3.2(c), $P(\log(1/\tilde{\tau}^2)) = 1/\tilde{\tau}$. In particular, P is not constant. P is univalent because for any compact subset of Π_∞ and ℓ_m large enough, P_{ℓ_m} is univalent on this set, which is evident from its definition. Let us define $x_0^- := \lim_{x \rightarrow -\infty} P(x)$. Since $(P)^{-1}$ is increasing to the right of x_0^- and $x_{\ell_m} < P_{\ell_m}(x)$ for every m and real negative x , we must have $x_0 \leq x_0^-$. Let us also note that $x_0^- < P(\log(1/\tau^2)) = 1/\tau$.

Precisely as in the covering maps case, we show that, for any $b < \tau^2$, the image of the half-plane $\{w : \Re w < \log b\}$ by the limit map P is contained in $\mathcal{D}((x_0, b'), \pi/2)$ where $b' = P(\log b)$.

We can now define a limit mapping ϕ which will be shown to belong to the \mathcal{EWF} class.

Fix any $\tau x_0 < R < \tau^2$ and define $\Pi_* = \{w : \Re w < \log R\}$. Consider P on Π_* . We set $U = P(\Pi_*)$. Then $\phi|_U := \exp \circ (P)^{-1}$. The intersection of U and the real axis is an interval (x_0^-, R') , where $R' = P(\log R)$.

We have shown that ϕ_{ℓ_m} converge to ϕ uniformly on any compact subset of $(x_0^-, R']$.

As $[\log(1/\tau^2), 0] \subset \Pi_*$ and $P(0) = 0$, $P(\log(1/\tau^2)) = 1/\tau$, there is a complex neighborhood S_1 of the interval $[1/\tau, 0]$, such that $\phi_{\ell_m} \rightarrow \phi$ uniformly in S_1 . In particular, $x_0 \leq x_0^- < 1/\tau < 0$ and ϕ is analytic in S_1 .

Recall that $\tilde{G}_{\ell_m}(z) = \tau_m^{-1} \phi_{\ell_m}(\tau_m^{-1} z)$. We define $\tilde{G}(z) = \lim_{m \rightarrow \infty} \tilde{G}_{\ell_m}(z)$ wherever the limit exists.

Repeating the proof for the covering maps, we extend ϕ to an analytic map which is defined in a complex neighborhood S_3 of the interval $[0, 1]$, and \tilde{G} is defined and analytic in a complex neighborhood S_4 of $[x_0, 0]$. In particular, $\tilde{G}(x_0) = x_0$, and x_0 is topologically non-repelling: $|\tilde{G}(x) - x_0| \leq |x - x_0|$ for every $x \in [x_0, 0]$. Then we have:

Lemma 3.5 $\tilde{G}([x_0, x_0^-]) = [x_0, x_0^-]$.

\tilde{G}_{ℓ_m} converge to \tilde{G} uniformly on a complex neighborhood of $[x_0, 0]$. We use now Lemma 3.2 (d). Since $\tilde{G}'_{\ell_m}(x_{\ell_m}) = \sqrt[m]{\tilde{\tau}_m^{-2}}$, the convergence implies $\tilde{G}'(x_0) = 1$. Coupled with the information that x_0 is topologically non-repelling on both sides, this implies the power-series expansion:

$$G(z) - x_0 = (z - x_0) + a(z - x_0)^{q+1} + O(|z - x_0|^{q+1})$$

with some $a \leq 0$ and some q even. First, we prove that $a \neq 0$, i.e. G is not the identity. If $G(z) = z$, then, for every x near 0, $\phi(x) = \phi(G(x)) = \phi(x)/\tau^2$, i.e. $\phi(x) = 0$, a contradiction. Thus, $a < 0$.

Now we prove that $q = 2$ considering a perturbation. There is a fixed complex neighborhood W of x_0 , such that the sequence of maps $(\tilde{G}_{\ell_m})^{-1}$ are well-defined in W and converges uniformly in W to \tilde{G}^{-1} . Since each ϕ_{ℓ_m} belongs to the Epstein class, then each $(\tilde{G}_{\ell_m})^{-1}$ extends to a univalent map of the upper (and lower) half-plane into itself. It extends also continuously on the real line, and has there exactly one fixed point, which is x_{ℓ_m} and which is repelling. Therefore, by the Wolff-Denjoy theorem, $(\tilde{G}^2)^{-1}$ has at most one fixed point in either half-plane, and one which is strictly attracting. This implies $q = 2$ by Rouché's principle.

Finally, we show that $x_0 = x_0^-$. Indeed, otherwise x_0^- would be a fixed point of \tilde{G} to the right of x_0 . Then, for big m , \tilde{G}_{ℓ_m} would have either two fixed points in the upper half-plane or a second real fixed point, a contradiction.

4 Dynamics of EWF Maps

4.1 Three levels of dynamics.

There are three types of dynamics which can be associated with a map ϕ from the \mathcal{EWF} -class: dynamics of \mathcal{H} on the circle, its extension to the complex dynamics similar to a polynomial-like map, and a so-called presentation function. Dynamics on the circle is easy to construct and was already described when we defined \mathcal{EWF} -class.

Construction of the complex dynamics H . The complex dynamical system will consists of two branches mapping onto the same range $\mathcal{D}(x_0, \tau x_0) \setminus \{0\}$.

The first one is ϕ . Its domain Ω_- is going to be contained in the domain U introduced in the definition of class \mathcal{EWF} . In particular it is contained in $\mathcal{D}(x_0, x_0/\tau)$ and, obviously, in $\mathcal{D}(x_0, \tau x_0)$.

The second branch is ϕ_{-1} mapping onto the same range. Its domain Ω_+ by rescaling is contained in τU and hence in $\mathcal{D}(x_0, \tau x_0)$. However, we want to emphasize that even though $\Omega_+ \cap \mathbb{R} = (x_0/\tau, x_0\tau)$, Ω_+ has no reason to be contained in $\mathcal{D}(x_0/\tau, x_0\tau)$. We have the following lemma, though:

Lemma 4.1

$$\overline{\Omega_-} \cap \overline{\Omega_+} = \left\{ \frac{x_0}{\tau} \right\}.$$

Proof. We will prove that

$$\overline{\phi^{-1}(\partial\mathcal{D}(x_0, \tau x_0) \setminus \{0\})} \cap \overline{\phi_{-1}^{-1}(\partial\mathcal{D}(x_0, \tau x_0) \setminus \{0\})} = \left\{ \frac{x_0}{\tau} \right\}.$$

Suppose that z belongs to the intersection of the boundaries of

$$\overline{\phi^{-1}(\partial\mathcal{D}(x_0, \tau x_0) \setminus \{0\})}$$

and

$$\overline{\phi_{-1}^{-1}(\partial\mathcal{D}(x_0, \tau x_0) \setminus \{0\})}.$$

If $y = \phi_{-1}(z)$, then

$$y \in \partial\mathcal{D}(x_0, \tau x_0).$$

By Lemma 1.3, $\phi_{-2}(y) = \phi(z) \in \partial\mathcal{D}(x_0, \tau x_0)$. But the preimage of $\mathcal{D}(x_0, \tau x_0)$ under ϕ_{-2} is contained in

$$\mathcal{D}(\tau^2 x_0, \tau^2 \phi^{-1}(\frac{x_0}{\tau})) = \mathcal{D}(\tau^2 x_0, x_0)$$

by Definition 1.4. Hence, the only possibility is for y to be equal to x_0 . In other words, z must be a preimage of x_0 by ϕ_{-1} .

Again by item 3 of Definition 1.4 such preimages belong to the closure of $\mathcal{D}(\frac{x_0}{\tau}, \tau x_0)$ which intersects the closure of

$$\overline{\phi^{-1}(\partial\mathcal{D}(x_0, \tau x_0) \setminus \{0\})} \subset \mathcal{D}(x_0, \frac{x_0}{\tau})$$

only at $\frac{x_0}{\tau}$.

□

Dynamics on the circle. The circle is formed by restricting H to the interval $(x_0, \tau x_0)$ and identifying x_0 with τx_0 . Since $\phi(\tau^{-1}x_0) = \tau x_0$ and $\phi_{-1}(\tau^{-1}x_0) = x_0$ while $\phi(x_0) = 0 = \phi_{-1}(\tau x_0)$ this gives a degree 1 circle homeomorphism which is smooth except at $\tau^{-1}x_0$. The rotation number of the circle dynamics generated by any \mathcal{EWF} -map equals the golden mean by definition.

The presentation function. The presentation function Π is defined on the interval $[x_0, \tau x_0]$ as follows. For $x \leq x_0/\tau$, $\Pi(x) = \tau x$. For $x > x_0/\tau$, $\Pi(x) = \phi_{-1}(x)$. Note that Π is not continuously defined on the circle.

4.2 Dynamics of the real presentation function

The presentation function is useful in the study of the dynamics of H because it is much simpler dynamically, but contains the full information about the post-critical set of H . Unlike H , Π is post-critically finite, since $\Pi(x_0\tau) = 0$ is the repelling fixed point of Π . Note also that the points $1, \tau^{-1}$ form a periodic orbit under Π . This will be used later to construct the conjugacy between presentation functions, but in this section we limit ourselves to the real dynamics of presentation functions.

As usual for post-critically finite maps, we get the following:

Lemma 4.2 *The first return map of Π into $(x_0, x_0/\tau^2)$ is defined everywhere except for a countable set and consists of diffeomorphic branches with negative Schwarzian derivative, all extendable to a fixed interval which contains $[x_0, x_0\tau^2]$ in its interior.*

Proof. By $\phi_{-1}\tau = \Pi^2$ interval $[x_0, x_0/\tau^2]$ is mapped onto $[x_0, 0]$ and extendable to $(\tau, 0)$ as a diffeomorphism. The first return map is obtained by composing this map piecewise with iterates of $\Pi^2 = \tau^2$. This results in a map defined except on a countable set with all branch extendable dynamically to $(\tau, 0)$. These extended branches are all compositions of ϕ_{-1} and τ and thus have negative Schwarzian derivative by the setup of the \mathcal{EWF} -class.

□

Lemma 4.3 *Preimages of 1 under Π are dense in $[x_0, \tau x_0]$.*

Proof. From Lemma 4.2, preimages of any point in $[x_0, x_0/\tau^2]$ by iterates of Π are dense in that interval. But $[x_0, x_0/\tau^2]$ is a fundamental domain for the dynamics: every orbit under Π passes through that interval. Hence, the preimages of any point of $[x_0, x_0/\tau^2]$ are dense in $[x_0, \tau x_0]$. That includes preimages of point $\tau^{-1} = \Pi^{-1}(1)$.

□

The connection of Π with H is summarized by this lemma.

Lemma 4.4 *A point $x \in (x_0, \tau x_0)$ is equal to $H^j(0)$, $j > 0$, if and only if $\Pi^k(x) = 1$ for some $k \geq 0$.*

Proof. In this proof we adopt the following notation which is consistent with a later use for towers: $\phi_k = \tau^{-k}\phi\tau^k$ for $k \in \mathbb{Z}$. We can always write for $j > 0$:

$$H^j(0) = \phi_{\epsilon_j} \circ \cdots \circ \phi_{\epsilon_1}(0)$$

for some $\epsilon_m = 0, -1$ for $m = 1, \dots, j$. By Fibonacci combinatorics, this sequence does not contain two -1 's in a row.

Assume first the $H^j(0) < x_0/\tau$. Then

$$\Pi(H^j(0)) = \tau H^j(0) = \phi_{\epsilon_{j-1}} \circ \cdots \circ \phi_{\epsilon_1-1}(0) = \phi_{\epsilon'_k} \circ \cdots \phi_{\epsilon'_1}(0) = H^k(0)$$

where the final representation is obtained from the functional equation (1) in the form $\phi_{-2} \circ \phi_{-1} = \phi_0$. Combinatorially, the whole process is equivalent to taking the sequence $(\epsilon_m)_{m=1}^j$ and whenever $\epsilon_m = 0$ and $\epsilon_{m+1} = -1$ replacing it with 0, while any occurrence of $\epsilon_m = 0$ which is not followed by $\epsilon_{m+1} = -1$ is replaced with -1 . Observe that $k \leq j$ and $j-k$ is equal to the number of occurrences of -1 in the sequence $(\epsilon_m)_{m=1}^j$. If $H^j(0) > x_0/\tau$, then $\Pi^2(H^j(0)) = \Pi(H^{j+1}(0))$ where $H^{j+1}(0) < x_0/\tau$. Since $\epsilon_{j+1} = -1$, by applying the previous case we see that $\Pi^2(H^j(0)) = H^k(0)$ where $k \leq j$ and again $j-k$ equals the number of occurrences of -1 in the sequence $(\epsilon_m)_{m=1}^j$.

We conclude that the first or the second iteration of Π maps $H^j(0)$ to $H^k(0)$ where $k < j$ unless the sequence (ϵ_m) consists only of zeros. That is only possible for $j = 1$ and thus every $H^j(0)$ is a preimage of $H(0) = 1$ under the iterates of Π .

Conversely, $\Pi^{-1}(H^j(0))$ could either be $\tau^{-1}H^j(0) = H_1^j(0)$ or $\phi_{-1}^{-1}H^j(0)$. In the first case, H_1 has two branches, one of which is ϕ_0 and another $\phi_1 = \phi_{-1} \circ \phi_0$ by the functional equation (1). Consequently, in this case $\Pi^{-1}(H^j(0)) = H^k(0)$ for $k > j$. In the second case $\Pi^{-1}(H^j(0)) = H^{j-1}(0)$. Hence, it follows by induction that every preimage of $1 = H(0)$ by Π is $H^k(0)$ for some k .

□

It follows that the circle dynamics of H has a dense orbit $\{H^j(0)\}_{j=0}^\infty$. This orbit is equal to $\{\Pi^{-k}(1) : k \geq 0\}$ by Lemma 4.4 and hence dense by Lemma 4.3. This yields:

Proposition 3 *The circle dynamics of H is conjugate to the linear rotation.*

Lemma 4.5 *Suppose that Π and $\hat{\Pi}$ are presentation functions for H and \hat{H} , respectively, both generated by maps from the \mathcal{EWF} -class. Then there is a unique order-preserving topological conjugacy v between them which fixes 1. This conjugacy is equal to the topological conjugacy between the corresponding circle maps.*

Proof. Let us first show that the conjugacy v between the circle mappings also conjugates between the presentation functions. On a dense subset $H^j(0)$, v is determined by $v(H^j(0)) = \hat{H}^j(0)$ for $j > 0$. But by considerations of the proof of Lemma 4.4, $\Pi(H^j(0)) = H^k(0)$ and $\hat{\Pi}(\hat{H}^j(0)) = \hat{H}^k(0)$ where k is the same in both cases. Then

$$v \circ \Pi(H^j(0)) = v(H^k(0)) = \hat{H}^k(0) = \hat{\Pi}(\hat{H}^j(0)) = \hat{\Pi}v(H^j(0)) .$$

Since it holds on a dense set, v conjugates Π to $\hat{\Pi}$.

For the uniqueness, observe that any conjugacy between the presentation functions normalized by $H(1) = 1$ maps the set $H^{-k}(1)$ onto $\hat{H}^{-k}(0)$ for $k \geq 0$. For an order preserving conjugacy there is a unique such mapping. By Lemma 4.4 it follows that the conjugacy is uniquely determined on a dense set. \square

4.3 Properties of orbits under complex dynamics.

Lemma 4.6 *For n even and non-negative, define u_n to be $\phi^{-1}(x_0\tau^{-n+1})$. For n odd and positive, define u_n to be $\phi_{-1}^{-1}(x_0\tau^{-n+1})$. For n even and non-negative, consider*

$$D_n = \mathcal{D}(x_0, u_n) \cup \mathcal{D}(u_{n+1}, \tau x_0) .$$

Suppose that z is in the domain of H and k is the smallest non-negative iterate for which $H^k(z) \in D_n$, for some even n . Then, there exists an inverse branch H^{-k} defined on the connected component of D_n which contains $H^k(z)$, which sends $H^k(z)$ to z .

Proof. Since the Poincaré neighborhood is simply connected, the only obstacle to constructing the inverse branch may be if the omitted value 0 is encountered. Thus suppose that for some $k' > 0$, ζ , which is an inverse branch of $H^{k-k'}$ well defined on the connected component D' of D_n which contains $H^k(z)$, maps $H^k(z)$ to $H^{k'}(z)$ and its image contains 0. First, consider the case when $D' = \mathcal{D}(x_0, u_n)$. Then $H^{k-k'+1}(0) \in (0, x_0\tau^{-n+1})$. Observe that the first entry of the iterates of 0 by H to the interval $(0, x_0\tau^{-n+1})$ occurs by a composition of branches of H , which is equal to the map $\tau^{-n} \circ \phi \circ \tau^n$, and the pullback of $(0, x_0\tau^{-n+1})$ by this map is the interval $(x_0\tau^{-n}, x_0\tau^{-n-1})$. Therefore, $\zeta((x_0, u_n)) \subset (x_0\tau^{-n}, x_0\tau^{-n-1})$. Using the property 4 of the definition of \mathcal{EWF} class, we get that $\zeta(\mathcal{D}(x_0, u_n)) \subset \mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1})$. In turn, $H^{-1}(\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1})) = \phi^{-1}(\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1})) \cup \phi_{-1}^{-1}(\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1}))$. Now, by the property 3 of the same definition,

$$\phi^{-1}(\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1})) \subset \phi^{-1}(\mathcal{D}(-x_0\tau^{-n+1}, x_0\tau^{-n+1})) \subset \mathcal{D}(x_0, u_n),$$

and

$$\begin{aligned}\phi_{-1}^{-1}(\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n-1})) &= \tau\phi^{-1}((\mathcal{D}(x_0\tau^{-n}, x_0\tau^{-n+1})) \subset \\ &\subset \tau\mathcal{D}(x_0, u_{n+2}) = \mathcal{D}(u_{n+1}, \tau x_0) .\end{aligned}$$

It means that $H^{k-k'-1}(z) \in D_n$, contrary to the hypothesis of the lemma. The remaining case $D' = \mathcal{D}(u_{n+1}, \tau x_0)$ is very similar. The first entry of the iterates of 0 by H to the $(x_0\tau^{-n}, 0)$ occurs by a composition of branches of H , which is equal to the map $\tau^{-n+1} \circ \phi \circ \tau^{n-1}$, and the pullback of $(0, x_0\tau^{-n+1})$ by this map is the interval $(x_0\tau^{-n-2}, x_0\tau^{-n-1})$. Therefore, $\zeta(\mathcal{D}(u_{n+1}, \tau x_0)) \subset \mathcal{D}(x_0\tau^{-n-2}, x_0\tau^{-n-1})$. In turn,

$$\phi^{-1}(\mathcal{D}(x_0\tau^{-n-2}, x_0\tau^{-n-1})) \subset \phi^{-1}(\mathcal{D}(-x_0\tau^{-n-1}, x_0\tau^{-n-1})) \subset \mathcal{D}(x_0, u_n),$$

and

$$\begin{aligned}\phi_{-1}^{-1}(\mathcal{D}(x_0\tau^{-n-2}, x_0\tau^{-n-1})) &= \tau\phi^{-1}((\mathcal{D}(x_0\tau^{-n-2}, x_0\tau^{-n-3})) \subset \\ \tau\phi^{-1}((\mathcal{D}(-x_0\tau^{-n-1}, x_0\tau^{-n-1})) &\subset \tau\mathcal{D}(x_0, u_{n+2}) = \mathcal{D}(u_{n+1}, \tau x_0).\end{aligned}$$

As in the first case, it is a contradiction.

□

Definition 4.1 Define $\Omega_{-,c}$ to be the range of the principal inverse branch of ϕ from the set $\mathcal{D}(x_0, x_0\tau) \setminus \mathbb{R}_-$. Similarly, $\Omega_{+,c}$ is the range of the principal inverse branch of ϕ_{-1} from $\mathcal{D}(x_0, \tau x_0) \setminus \mathbb{R}_+$. Equivalently, $\Omega_{-,c}$ is the preimage of the strip $|\Im w| < \pi$ by $\log \phi$ and likewise $\Omega_{+,c}$ is the preimage of the same strip by $\log \phi_{-1}$.

Lemma 4.7 Take a point z with an infinite orbit under H . Moreover, for each $k \geq 0$, $H^k(z) \in \Omega_{-,c} \cup \Omega_{+,c}$. Also, suppose that the distance from the set $\{H^k(z) : k \geq 0\}$ to \mathbb{R} is 0. Then, $z \in \mathbb{R}$.

Proof. By Proposition 3 the dynamics of H on the interval $[x_0, \tau x_0]$ is transitive and so if the orbit of z accumulates on it, then for a subsequence $H^{k_j}(z) \rightarrow x_0$. Recall Lemma 4.6 and in particular points u_n defined there. By Lemma 1.2 an intersection of Ω_- with $D(x_0, \epsilon)$ for a small ϵ is contained in the set $\mathcal{D}(x_0, u_n)$ where n depends on ϵ and may be made tend to infinity as ϵ tends to 0. As a consequence, there is a sequence $n(j)$ tending to ∞ such that $H^{k_j}(z) \in D_{n(j)}$. Then by Lemma 4.6 an inverse branch of H^{k_j} exists which tracks back the orbit of z . Since the orbit z stays in the set $\Omega_{-,c} \cup \Omega_{+,c}$, where the only preimage of \mathbb{R} is inside \mathbb{R} , this inverse branch fixes the real line. As the consequence of the Epstein property postulated for the \mathcal{EWF} -class, the orbit is confined to set $\mathcal{D}(H^{-k}((x_0, u_{n(j)})))$ where the preimages are taken by the real dynamics. Since the real dynamics is conjugated to the golden mean circle rotation by Proposition 3, the lengths of the intervals $H^{-k}((x_0, u_{n(j)}))$ tend to 0 with j uniformly with respect to k . Thus, z is in the intersection of the sequence of disks with radii tending to 0 and centers on \mathbb{R} .

□

5 Quasiconformal equivalence

5.1 Tempering.

Given the mappings ϕ and $\hat{\phi}$ from the \mathcal{EWF} class, we would like to conjugate between their complex dynamics by a quasiconformal map. For technical reason, such a conjugation is difficult to obtain between the corresponding complex extensions H and \hat{H} . We will use modified extensions which differ by the range, which is made smaller.

The range of *tempered* H will be contained in $\mathcal{D}(x_0, \tau x_0)$, but for technical reasons having to do with quasiconformal constructions, we prefer the border to intersect the real line at angles less than $\pi/2$. To this end, choose $\frac{\pi}{2} > \beta_l > \beta_r > \frac{\pi}{4}$ where β_r is close to $\pi/2$ and will be specified shortly. Let A_l denote the angle $\{z : |\arg(z - x_0)| < \beta_l\}$ and similarly $A_r = \{z : |\arg(z - \tau x_0) - \pi| < \beta_r\}$.

Then

$$V' = \mathcal{D}(x_0, x_0\tau) \cap A_l \cap A_r, V := V' \setminus \{0\}.$$

The left branch of H is ϕ restricted to the the preimage of V , denoted U_- . The right branch is the preimage of V by ϕ_{-1} and its domain will be called U_+ . Since $V' \subset D(0, |\tau x_0|)$, then from the definition of the \mathcal{EWF} -class, $U_- \subset \mathcal{D}(x_0, x_0/\tau)$.

H extends analytically through the boundary of U_- at any point with the exception of x_0 . At x_0 , we refer to Lemma 1.2. It implies that for β_r close enough to $\frac{\pi}{2}$, $U_- \subset U \subset V'$. In particular, since $\Omega_- \subset U$, the range V' of the tempered map still contains Ω_- .

To analyze U_+ , we need to view ϕ_{-1} as a rescaled version of ϕ . Because of that, $U_+ \subset \tau U \subset \mathcal{D}(x_0, x_0\tau)$. However, additionally we have Lemma 1.2 whose statement can be applied to τU and means that locally U_+ fits the angle A_r . On the other side, $U_+ \cap \mathbb{R} = (x_0/\tau, x_0\tau)$ so that it avoids a neighborhood of x_0 . It follows that if β_r is close enough to $\pi/2$, then $U_+ \subset V'$. It also follows that $\Omega_+ \subset V'$.

Finally, $\overline{U_-} \cap \overline{U_+} = \{x_0\tau^{-1}\}$ as the consequence of Lemma 4.1. Additionally, as a corollary from the construction,

$$V' \supset \Omega_- \cup \Omega_+. \quad (19)$$

Theorem 6 *Suppose that $\phi, \hat{\phi} \in \mathcal{EWF}$ and let H, \hat{H} be their tempered complex dynamics, defined on affinely similar ranges V', \hat{V}' . Then, there is a quasiconformal automorphism Ψ of the plane, symmetric w.r.t. \mathbb{R} and fixing 0, 1 which conjugates tempered H and \hat{H} on their respective domains $U_- \cup U_+, \hat{U}_- \cup \hat{U}_+$.*

5.2 Conjugacy between presentation functions.

If two maps from \mathcal{EWF} class are given, then by Lemma 4.5, the conjugacy between their circle dynamics is equal to the conjugacy between their real presentation

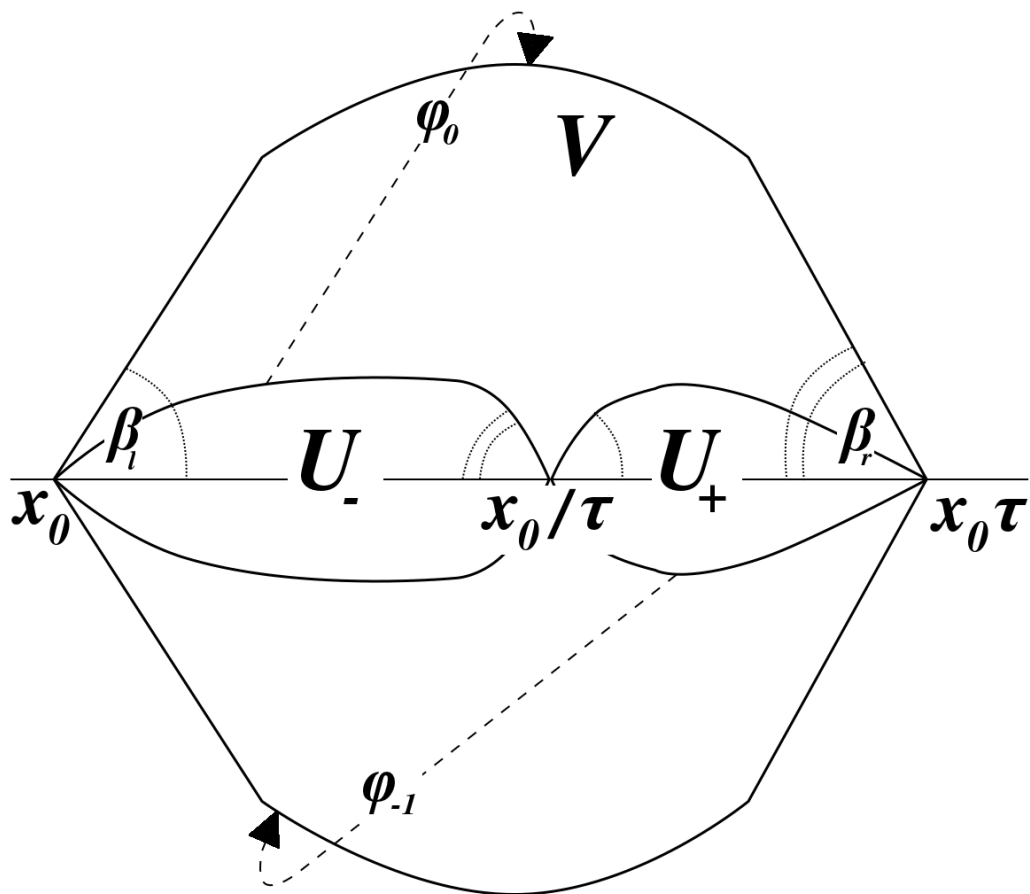


Figure 3: The tempered mapping.

functions. This is the overall strategy of the proof of Theorem 6. We will first construct the conjugacy between the presentation functions.

Complex presentation function Π . Recall the definition of the presentation function on the segment $[x_0, \tau x_0]$. If $x \in [x_0, \tau^{-1}x_0]$, set $\Pi(x) = \tau x$. If $x \in [\tau^{-1}x_0, \tau x_0]$, set $\Pi(x) = \phi_{-1}(x)$.

Π can also be extended to a complex map with range V' . The linear branch maps $W_- := \tau^{-1}V'$ onto V' . Because $\beta_r < \beta_l$, we get $W_- \subset V'$. The non-linear branch will map only onto $W_- \setminus \{0\}$. Let W_+ be $\phi_{-1}^{-1}(W_- \setminus \{0\})$. Taking into account the rescaling,

$$W_+ = \tau \phi^{-1}(\tau^{-2}V) .$$

Since $\tau^{-2}V \in D(0, x_0/\tau)$, by the defining properties of class \mathcal{EWF} ,

$$\phi^{-1}(\tau^{-2}V) \subset \mathcal{D}(x_0, x_0/\tau^2) .$$

Finally $W_+ \subset \mathcal{D}(x_0/\tau, x_0\tau)$. Invoking Lemma 1.2, we see that for β_r close enough to $\pi/2$, $W_+ \subset V'$. Moreover, $\overline{W}_+ \cap \overline{W}_- = \{\frac{x_0}{\tau}\}$.

Notational convention. We suppose that ϕ and $\hat{\phi}$ are given as in the statement of Theorem 6. Moreover, all objects constructed from $\hat{\phi}$ (complex dynamics, presentation function, etc) will also be marked by $\hat{\cdot}$.

Technical tools. By an *arc* we will mean a homeomorphic image of either an open interval or a circle. The arc is term *quasiconformal* if the homeomorphism can be extended to a quasiconformal homeomorphism of the plane.

Definition 5.1 *Let w be an arc and $y \in w$. We say that y is sectorially accessible iff there is $\epsilon > 0$ and a pair of vertical angles of positive measure with the vertex at y and interior denoted with A so that $w \cap A \cap D(y, \epsilon) = \emptyset$.*

Fact 5.1 *Let w be an arc and $Y = \{y_1, \dots, y_n\}$ a finite collection of sectorially accessible points of w . Then, if each connected component of $w \setminus Y$ is a quasiconformal arc, then w is quasiconformal as well.*

Definition 5.2 *Now, suppose that h is an orientation-preserving real homeomorphism of an open interval in \mathbb{R} (perhaps unbounded) onto its image. If y is a point in the domain of h , then h is said to be quasi-symmetric at y provided that there are $\epsilon > 0$ and K so that whenever $0 < |\eta| < \epsilon$, then*

$$\frac{|h(y + \eta) - h(y)|}{|h(y - \eta) - h(y)|} \leq K .$$

Fact 5.2 *If h is a real orientation-preserving homeomorphism of an open possibly unbounded interval I in \mathbb{R} onto its image, $Y = \{y_1, \dots, y_n\}$ is a finite collection of points from I , h is quasi-symmetric at each point from Y and h is quasi-symmetric on each connected component of $I \setminus Y$, then h is quasi-symmetric on its entire domain.*

Initial pre-conjugacy.

Proposition 4 *There exists a quasiconformal mapping Ψ_1 of the plane, symmetric with respect to \mathbb{R} , affine outside of V' , which satisfies the conjugacy condition*

$$\Psi_1 \circ \Pi(z) = \hat{\Pi} \circ \Psi_1(z)$$

for $z \in \partial W_- \cup \partial W_+$.

Let A denote the real and orientation-preserving affine map which transforms V' onto \hat{V}' . Then A' is a quasiconformal map equal to A outside of V' and fixing 0. Then, on W_- consider the mapping $A_- = \hat{\tau}^{-1} \circ A' \circ \tau$. A_- is quasiconformal on W_- and fixes 0. Then, A_+ is defined on W_+ as the lifting of A_- to the universal covers $\phi_{-1}, \hat{\phi}_{-1}$. A_+ is also quasiconformal.

Consider also the Jordan arc w which consists of the boundary arcs of V' , W_- and W_+ intersected with $\overline{\mathbb{H}}_+$. \hat{w} is analogous. Additionally, a homeomorphism u of \mathbb{R} onto itself has been defined, which consists of A outside of $(x_0, \tau x_0)$, is equal to A_- on $(x_0, \tau^{-1}x_0)$ and to A_+ on $(\tau^{-1}x_0, \tau x_0)$.

Lemma 5.1 *w, \hat{w} are quasiconformal.*

Proof. We will rely on Fact 5.1 and only give a proof for w . Points $x_0, \tau^{-1}x_0, \tau x_0$ are sectorially accessible by the choice of angles in the construction of tempered dynamics and Lemma 1.2. The boundary arcs of V' and W_- are clearly quasiconformal. $\log \phi_{-1}$ is the Fatou coordinate of rescaled G . Then, the boundary of W_+ is the preimage by the Fatou coordinate of the $2\pi i$ -periodic curve which is the image of the boundary of W_- under the log. This curve is quasiconformal and so is its preimage based on Fact 1.3. □

Lemma 5.2 *The mapping u is quasi-symmetric on \mathbb{R} .*

Proof. This will be based on Fact 5.2. After removing points $x_0, \tau^{-1}x_0, \tau x_0$, u is quasi-symmetric on each connected component. This is obvious except on $(\tau^{-1}x_0, \tau x_0)$, where one can invoke Fact 1.3. The quasi-symmetry at x_0 and $\tau^{-1}x_0$ is also clear, so the mapping is analytic on one-sided neighborhoods of those points. It remains to consider the quasi-symmetry at τx_0 .

Let G_{-1} denote $\tau G \tau^{-1}$. Denote $y_n = G_{-1}^n(\tau^{-1}x_0)$, likewise in the “hatted” space. The quasi-symmetry at τx_0 follows if we can show that

1.

$$\exists K > 1 \forall n \geq 0 \ K^{-1} \leq \frac{\tau x_0 - y_n}{\hat{\tau} \hat{x}_0 - \hat{y}_n} \leq K ,$$

2.

$$\forall n \geq 0 \ \frac{\tau x_0 - y_n}{\tau x_0 - y_{n+1}} \leq K$$

and the same holds in the “hatted” space.

The second statement is obvious, since the ratios tend to 1 by the dynamics of a neutral point of G_{-1} . The first statement follows from the form of the Fatou coordinate, namely that it is $\frac{a}{(y-\tau x_0)^2}$ followed by a map whose distance from the identity is bounded. Also, $\log A_-$ moves points by bounded distances.

□

Construction of Ψ_1 . The union of A , A_- and A_+ and u already defines Ψ_1 in the closure of \mathbb{H}_+ except for the quasiconformal disk bounded by w . We can also extend the definition to the lower half-plane using the Beurling-Ahlfors theorem, see [2], based on Lemma 5.2. Then, extension for the region bounded by w is achieved by a quasiconformal reflection based on Lemma 1.1. Finally, Ψ_1 is defined in the lower-half plane by reflection from \mathbb{H}_+ . The properties claimed for it in Proposition 4 are clear from the construction, so the proof of this Proposition is finished.

Conjugacy between presentation functions.

Proposition 5 *There exists a quasiconformal mapping Ψ_2 of the plane, symmetric with respect to \mathbb{R} , affine outside of V' , which satisfies the conjugacy condition*

$$\Psi_2 \circ \Pi(z) = \hat{\Pi} \circ \Psi_2(z)$$

for $z \in \partial W_- \cup \partial W_+$ and for $z \in (x_0, \tau x_0)$.

To construct Ψ_2 , we first modify Ψ_1 by imposing additional conditions $\Psi_1(0) = 0, \Psi_1(1) = 1, \Psi_1(\tau^{-1}) = \hat{\tau}^{-1}$ without losing the properties claimed in Proposition 4. This is easy to do, since we can move finitely many points in side W_-, W_+ by quasiconformal mappings which remain identities on the boundary. Dynamically these conditions mean that Ψ_1 conjugates on the post-critical set of the presentation function. We can then construct a sequence of pull-backs Ψ_1^n , with $\Psi_1^0 = \Psi_1$ and $\hat{\Pi} \circ \Psi_1^{n+1} = \Psi_1^n \circ \Pi$, defined uniquely by the condition that they map the real domain onto itself preserving the orientation.

By construction, Ψ_1^n maps all preimages of 1 by Π of order not exceeding n onto the corresponding preimages of 1 by $\hat{\Pi}$. We can define Ψ_2 as any limit of the sequence $\Psi_1^{n_k}$ using normality of quasiconformal mappings with the same maximal dilatation. Then, Ψ_2 is the same as Ψ_1 on the outside of the domain of Π and additionally maps all preimages of 1 by Π onto the corresponding preimages of 1 by $\hat{\Pi}$. By Lemma 4.3 this implies conjugacy on the entire real domain $(x_0, \tau x_0)$. Proposition 5 has been proved.

5.3 Conjugacy between tempered complex dynamics of H, \hat{H} .

Construction of the pre-conjugacy.

Proposition 6 *There exists a quasiconformal mapping Ψ_3 of the plane, symmetric with respect to \mathbb{R} , affine outside of V' , which satisfies the conjugacy condition*

$$\Psi_3 \circ H(z) = \hat{H} \circ \Psi_3(z)$$

for $z \in \partial U_- \cup \partial U_+$ and for $z \in (x_0, \tau x_0)$.

Start by defining maps B_{\pm} . Let B_- be defined on U_- as the lifting of Ψ_2 by the universal covers $\phi, \hat{\phi}$. Since Ψ_2 was the conjugacy between real presentation functions of $(x_0, \tau x_0)$ and by Lemma 4.5 also the conjugacy between the real dynamics of H, \hat{H} , B_- remains the same as Ψ_2 on $(x_0, \tau^{-1}x_0)$. Additionally, it satisfies the conjugacy condition on the boundary. Map B_+ is defined similarly on U_+ as the lifting of Ψ_2 to the universal covers $\phi_{-1}, \hat{\phi}_{-1}$. By the same arguments, it is equal to Ψ_2 on the real trace and satisfies the conjugacy condition on the boundary.

We can extend the union of B_- and B_+ restricted to \mathbb{H}_+ to the lower half-plane by Ψ_2 and to the complement of V' in the upper half plane also by Ψ_2 . To complete the proof of Proposition 6, it remains to extend the mapping to the set $V' \cap \mathbb{H}_+ \setminus (U_- \cup U_+)$.

This follows by the same method as in the proof of Proposition 4. We first consider the Jordan arc v which consists of the boundary arcs of V', U_-, U_+ in the upper half plane and the analogous arc \hat{v} .

Lemma 5.3 *Arcs v, \hat{v} are quasiconformal.*

Proof. The proof is very similar to the proof of Lemma 5.1 and omitted.

□

Now Ψ_3 is constructed on the region bounded by v by the quasiconformal reflection and finally extended to the lower half-plane by reflection.

Proof of Theorem 6. Thus, we construct a sequence of quasi-conformal homeomorphisms Ψ^n of the plane, by setting $\Psi^0 = \Psi_3$ and defining Ψ^n for $n > 0$ as Ψ^{n-1} outside of $U_+ \cup U_-$ and to be the lifting of Ψ^{n-1} to the universal covers $H|_{U_+}, \hat{H}|_{\hat{U}_+}$ and $H|_{U_-}, \hat{H}|_{\hat{U}_-}$. Both liftings are uniquely defined by the requirement that Ψ^n should fix the real line with its orientation.

The sequence $\Psi^n(z)$ actually stabilizes for every $z \notin \tilde{K}_H$. By Lemma 6.2, $\tilde{K}_H = K_H$. So Ψ^n converge on the complement of K_H and by taking a subsequence can be made to converge globally to some map Ψ^∞ . Outside of K_H , Ψ^∞ satisfies the functional equation $\Psi^\infty H = \hat{H} \Psi^\infty$ and then it also satisfies it on K_H by continuity, since K_H has an empty interior by Lemma 6.1. So we can set $\Psi := \Psi^\infty$ and this concludes the proof of Theorem 6.

6 Julia sets and Towers

6.1 Julia sets.

We consider the filled-in Julia set for the untempered dynamics H . It can be constructed in several stages. First $K'_H = \{z : \Omega_- \cup \Omega_+ : \forall j > 0 H^j(z) \in \Omega_- \cup \Omega_+\}$. Then, $K''_H = \{x_0, \tau x_0\}$. Then, $K'''_H = \{x \in \Omega_- \cup \Omega_+ : \exists j \geq 0 H^j(x) = x_0/\tau\}$. By definition, $K_H = K'_H \cup K''_H \cup K'''_H$.

At the end of this section we will prove the following theorem:

Theorem 7

$$K_H = \overline{\{H^{-j}(x_0/\tau) : j \geq 0\}}.$$

As a corollary, we get

Lemma 6.1 *Set K_H has an empty interior.*

Proof. If the interior of K_H is not empty, then by Theorem 7, the interior of K_H contains a neighborhood of x_0/τ . But any neighborhood of that point sticks out of the domain of definition for H . □

The Julia set for tempered dynamics. By analogy to the set K_H , we consider the filled-in Julia set \tilde{K}_H for the tempered dynamics. It can be constructed in several stages. First $\tilde{K}'_H = \{z : U_- \cup U_+ : \forall j > 0 H^j(z) \in U_- \cup U_+\}$. Then, $\tilde{K}''_H = \{x_0, \tau x_0\}$. Then, $\tilde{K}'''_H = \{x \in U_- \cup U_+ : \exists j \geq 0 H^j(x) = x_0/\tau\}$. By definition, $\tilde{K}_H = \tilde{K}'_H \cup \tilde{K}''_H \cup \tilde{K}'''_H$.

Lemma 6.2 *For any dynamics generated by the \mathcal{EWF} class, $\tilde{K}_H = K_H$.*

Proof. Since the tempered dynamics is a restriction of the untempered version to a smaller domain, $\tilde{K}_H \subset K_H$. In order to prove the opposite inclusion it will be enough to show that $K_H \subset U_- \cup U_+$. This is because K_H is invariant. Recall that H maps any point of $\Omega_- \cup \Omega_+$ not in $U_- \cup U_+$ into $\mathcal{D}(x_0, \tau x_0) \setminus V'$. By inclusion (19) this is outside of the domain of untempered dynamics and hence such points cannot belong to K_H . □

6.2 Dynamics in towers.

Let H be the complex dynamics, tempered or not, generated by some ϕ in the \mathcal{EWF} -class.

Definition 6.1 Define, for $n = 0, 1, 2, \dots$, $H_{-n}(z) = \tau^n H(z/\tau^n)$. Then $\tau^n K_H$ is the Julia set of the map $H_{-n} : \Omega_{-n} \rightarrow D_{-n}^*$, where $\Omega_{-n} = \tau^n(\Omega_- \cup \Omega_+)$, $D_{-n}^* = \tau^n \mathcal{D}(x_0, \tau x_0) \setminus \{0\}$.

The collection of maps $H_n : U_n \rightarrow V_n$, $n = 0, -1, \dots$ forms the tower of H , tempered or untempered, respectively. Map H_n will be referred to as the n -th level of the tower.

Lemma 6.3 For any $0 \geq m > n$, each branch of untempered H_m on its domain is a composition for branches of H_n . This includes an assertion that for each component of Ω_m there is a particular composition of branches on H_n which is well defined on this entire component.

Proof. By induction, it is enough to prove this statement when $n = m - 1$. By rescaling, we can reduce the situation to $m = 0, n = -1$.

Then, one branch of H_0 is ϕ_{-1} defined on Ω_+ . It is the same as a branch of H_{-1} defined on $\tau\Omega_-$. All we need to check is $\Omega_+ \subset \tau\Omega_-$. By definition, however, Ω_+ is the preimage by $\phi_{-1} = \tau\phi\tau^{-1}$ of $\mathcal{D}^*(x_0, \tau x_0)$ which is $\tau\phi^{-1}(\mathcal{D}^*(x_0, \tau^{-1}x_0))$, which is clearly inside $\tau\Omega_-$.

The second branch of H_0 is a composition $\phi_{-2} \circ \phi_{-1}$, both of which are branches on H_{-1} . We need, however, to check the inclusions between domains. ϕ_{-1} as a branch of H_{-1} is defined on $\tau\Omega_-$. So the next inclusion to check is $\Omega_- \subset \tau\Omega_-$.

This follows from a sequence of inclusions:

$$\begin{aligned} \Omega_- &= \phi^{-1}(\mathcal{D}^*(x_0, \tau x_0)) \supset \phi^{-1}(\mathcal{D}(x_0\tau^{-1}, \tau x_0)) = \phi^{-1}(\tau\mathcal{D}(x_0, x_0\tau^{-2})) \supset \\ &\supset \phi^{-1}(\tau\phi^{-1}(\mathcal{D}^*(x_0\tau^{-1}, x_0\tau^{-2}))) = (\phi^{-1} \circ \tau \circ \phi^{-1} \circ \tau^{-2})\mathcal{D}^*(x_0, \tau x_0). \end{aligned}$$

But

$$\phi^{-1} \circ \tau \circ \phi^{-1} \circ \tau^{-2} = \tau^{-1}\phi_{-1}^{-1} \circ \phi_{-2}^{-1} = \tau^{-1}\phi^{-1}$$

by Lemma 1.3. This proves $\Omega_- \supset \tau^{-1}\Omega_-$.

Next, we need to check that $\phi_{-1}(\Omega_-) = \tau\phi(\tau^{-1}\Omega_-) \subset \tau^2\Omega_-$, where the last set is the domain of ϕ_{-2} . This inclusion is equivalent to

$$\tau^{-1}\phi\tau^{-1}(\Omega_-) = G(\Omega_-) \subset \Omega_- .$$

The last inclusion follows from

$$\tau^{-2}\phi = \phi \circ G$$

which implies that G maps the domain of ϕ into itself. □

Lemma 6.4 *The assertion of Lemma 6.3 remains true for the dynamics on tempered domains if the parameter β_r in the definition of tempered domain is close enough to $\pi/2$. That is, H_n considered on a component $\tau^n U_\pm$ of its domain can be realized as a composition of branches of H_m , also defined on their tempered domains $\tau^m U_\pm$.*

Proof. We only need to check that inclusions between the domains stated in the proof of Lemma 6.3 remain true for tempered domains.

The first inclusion is $U_+ \subset \tau U_-$. U_+ was defined as the preimage by ϕ_{-1} of $\tau^{-1}V$. Therefore,

$$U_+ = \tau\phi^{-1}(\tau^{-2}V) \subset \tau\phi^{-1}(V) = \tau U_-$$

as needed.

Next, we need $U_- \subset \tau U_-$. The boundaries of the corresponding untempered domains intersect only at x_0 . In a neighborhood of this point τU_- is the preimage by a real conformal map of an angle \mathcal{A}_r . Since β_r is chosen greater than $\frac{\pi}{4}$ and in view of Lemma 1.2, the desired inclusion holds on a sufficiently small neighborhood of x_0 . Outside of this neighborhood, the distance between the boundaries of the untempered domains is positive. By picking β_r sufficiently close to $\pi/2$, we can move the border of τU_- arbitrarily little from the border of $\tau\Omega_-$ and so preserve the inclusion.

Thirdly, we need $U_+ = \phi_{-1}^{-1}(U_-) \subset \tau^2 U_-$. But we have already seen that $U_+ \subset \tau U_-$ and $U_- \subset \tau U_-$. After scaling by τ , the desired inclusion follows. □

Proposition 7 *For every $z \in \mathbb{C}$ which is never mapped to \mathbb{R} by the untempered tower dynamics, there exist sequences $z_n \in \mathbb{C}$ and $m_n \in \mathbb{N} \cup \{0\}$, $n = 0, 1, \dots$ chosen so that $z_0 = z$, z_{n-1} belongs to the domain of $H_{-m_{n-1}}$ and z_n is an image of $H_{-m_{n-1}}(z_{n-1})$ by the tower dynamics. Furthermore, for some $\eta > 0$ and every $n > 0$*

- $\text{dist}(z_n, \mathbb{R}) > \eta |\tau|^{m_n}$ with dist meaning the Euclidean distance, or
-

$$\tau^{-m_n} z_n \in (\Omega_- \cup \Omega_+) \setminus (\Omega_{+,c} \cup \Omega_{-,c})$$

(cf. Definition 4.1.)

For every z , one can find a sequence (n_p) and choose one case of the alternative, so that the claim of this Proposition holds for all $n := n_p$ with this particular case.

Auxiliary dynamics of Γ . For a point $\tau^n x_0$ the highest level tower map defined on a neighborhood of this point is H_{-n-2} . We have

$$H_{-n-2}(\tau^n x_0) = \tau^{n+2} H(\tau^{-2} x_0) \tau^{n+2} \tau^{-1} x_0 = \tau^{n+1} x_0. \quad (20)$$

This leads one to consider the map

$$\Gamma(z) = \tau^{-1} H_{-2}(z).$$

Then, formula (20) means that $\tau^n x_0$ is a fixed point for $\Gamma_{-n} = \tau^n \Gamma \tau^{-n}$, in particular, x_0 is a fixed point of Γ .

To find out the local dynamics at this point, observe that

$$\Gamma \circ \Gamma = \tau^{-1} H_{-2} \tau^{-1} H_{-2} \tau \tau^{-1} = \tau^{-1} H_{-2} H_{-1} \tau^{-1} = \tau^{-1} H \tau^{-1} = G.$$

However, Γ is orientation-reversing on the real line, so $\Gamma'(x_0) = -1$. As a consequence of the defining properties of the \mathcal{EWF} -class, x_0 is the global attractor for Γ^{-1} on $\mathbb{C} \setminus \mathbb{R}$.

Lemma 6.5 *Let Ψ_k denote the composition of k maps in the form $H_{-k-1} \circ \dots \circ H_{-2}$ defined on a neighborhood of x_0 . We put $\Psi_0 = \text{id}$. Choose $z \in \mathbb{C}$ so that $\arg(z - x_0) \bmod \pi \in [\frac{\pi}{5}, \frac{4\pi}{5}]$. There exists $\epsilon_0 > 0$ such that for every $\epsilon_0 \geq \epsilon > 0$ and z as above there is $k \geq 0$ so that $\arg(\Psi_k(z) - \tau^k x_0) \in [\frac{\pi}{5}, \frac{4\pi}{5}]$, $|\Psi_k(z) - \tau^k x_0| < 2\epsilon |\tau|^k$ and either $|\Psi_k(z) - \tau^k x_0| \geq |\tau|^k \epsilon$, or $\Psi_{k+1}(z)$ is defined and*

$$\arg(\Psi_{k+1}(z) - \tau^{k+1} x_0) \bmod \pi \in (\frac{\pi}{10}, \frac{\pi}{5}) \cup (\frac{4\pi}{5}, \frac{9\pi}{10}).$$

Proof. One easily shows by induction that, in a generalization of formula (20), $\tau^{-k} \Psi_k = \Gamma^k$. Then ϵ_0 should be chosen so that Γ is defined on $D(x_0, \epsilon_0)$.

Since Γ^{-1} attracts to x_0 on every angle disjoint from the real line, then the orbit $\Gamma^k(z)$ for z in this angle has to either leave the angle, or leave $D(x_0, \epsilon)$. Then choose k to the first moment when that happens. If $|\Gamma^k(z) - x_0| \geq \epsilon$, this immediately translates by rescaling to the first case of this Lemma. The upper bound $|\Gamma^k(z) - x_0| < 2\epsilon$ can be obtained by specifying ϵ_0 small enough, since $\Gamma'(x_0) = -1$.

If $\Gamma^k(z) - x_0$ is still inside the angle, then we can reason by the conformality of Γ at x_0 and, by decreasing ϵ_0 if needed, get that $\Gamma^{k+1}(z) - x_0$ is still in a slightly larger angle, leading to the second case in the Lemma.

□

Proof of Proposition 7 Suppose z_{n-1} has been chosen in the domain of $H_{-m_{n-1}}$. Let $z = H_{-m_{n-1}}(z_{n-1})$. The first possibility is that z_{n-1} is in the Julia set of $H_{m_{n-1}}$. This case is resolved by applying Lemma 4.7. If $H_{-m_{n-1}}^k(z)$ gets away from the real line, or outside of $\Omega_{-,c} \cup \Omega_{+,c}$ for any $k > 0$, then we can set $z_n = H_{-m_{n-1}}^k$ and $m_n := m_{n-1} + 1$. One of those must occur, since otherwise by Lemma 4.7, $z \in \mathbb{R}$ contrary to the hypothesis of Proposition 7.

If z_{n-1} is not in the Julia set, then we map $z = H_{-m_{n-1}}(z_{n-1})$ by the dynamics of $H_{-m_{n-1}}$ until the first moment q when $w := H_{-m_{n-1}}^q(z)$ is no longer in the domain of $H_{-m_{n-1}}$, or w gets away from the real line by $\eta|\tau|^{m_{n-1}}$. If the second possibility occurs, then we set $z_n = w$ and proceed like in the previous case. Otherwise, w is outside the domain of $H_{-m_{n-1}}$ and still close to the real line, thus in a neighborhood of one of the points where the domain of $H_{-m_{n-1}}$ touches \mathbb{R} , that is x_0 , τx_0 , and x_0/τ rescaled by $\tau^{m_{n-1}}$. The size of those neighborhoods can be controlled by setting η small enough.

In other words, $\tau^{-m}w \in D(x_0, \epsilon)$ where $m = m_{n-1}, m_{n-1} + 1$ or $m_{n-1} - 1$ and ϵ is small depending on η .

We then apply maps Ψ_k from Lemma 6.5 to $\tau^{-m}w$ obtaining the sequence $w_k = \tau^m \Psi_k(\tau^{-m}x)$.

The first possibility in Lemma 6.5 means that w_k is inside a fixed angle disjoint from \mathbb{R} in a distance greater than $\epsilon|\tau|^{m+k}$ from $x_0\tau^{m+k}$. So, its distance from \mathbb{R} is bounded away from 0 by $\eta|\tau|^{m+k}$, perhaps after decreasing η again. At the same time, its distance from $x_0\tau^{m+k}$ is also bounded above by $2\epsilon|\tau|^{m+k}$, so by choosing ϵ suitably small, w_k belong to the domain of H_{-m-k-2} . So, in this case we set $z_n = w_k$ and $m_n = m + k + 2$.

The second possibility is that $\arg(w_k - x_0\tau^{m+k}) \bmod \pi$ is either between $\pi/10$ and $\pi/5$, or between $4\pi/5$ and $9\pi/10$. The boundary of $\Omega_{-,c}$ is tangent to \mathbb{R} at x_0 and likewise the boundary of $\Omega_{+,c}$ is tangent to \mathbb{R} at τx_0 . Since on the other hand, the boundaries of Ω_+, Ω_- intersect \mathbb{R} at angle $\pi/4$ by choosing ϵ suitably small we can guarantee that

$$\tau^{-m-n}w_k \in (\Omega_- \cup \Omega_+) \setminus (\Omega_{-,c} \cup \Omega_{+,c})$$

leading to the second case in Proposition 7 with $z_n = w_k$ and $m_n = m + k$.

Proposition 7 follows.

6.3 Hyperbolic Metric

Definition 6.2 Let ρ_n be the Poincaré metric on $\mathcal{D}(\tau^n x_0, \tau^{n+1} x_0) \setminus \mathbb{R}$. Since this set is disconnected, it means the Poincaré metric on each connected component, with infinite distance between the components. Similarly, ρ_∞ is the Poincaré metric of $\mathbb{C} \setminus \mathbb{R}$.

Note that ρ_∞ is invariant under the rescaling $z \mapsto \tau z$.

If ρ is a metric and F a function, we will write $D_\rho F(z)$ for the expansion ratio with respect to the metric ρ , thus

$$D_\rho F(z) = \frac{|F^* d\rho(z)|}{|d\rho(z)|}.$$

By Schwarz's lemma, we have $D_{\rho_\infty} H(z) > 1$ for every $z \in \Omega_+ \cup \Omega_- \setminus H^{-1}(\mathbb{R})$. We will observe expansion of the hyperbolic metric based on the following fact:

Fact 6.1 *Let X and Y be hyperbolic regions and $Y \subset X$ and $z \in Y$. Let ρ_X and ρ_Y be the hyperbolic metrics of X and Y , respectively. Suppose that the hyperbolic distance in X from z to $X \setminus Y$ is no more than D . For every D there is $\lambda_0 > 1$ so that $|\iota'(z)|_H \leq \frac{1}{\lambda_0}$, where $\iota : Y \rightarrow X$ is the inclusion, and the derivative is taken with respect to the hyperbolic metrics in Y and X , respectively.*

The following lemma is stated in terms of H , but clearly it applies to any H_k as well, because the only difference is the conjugation by a power of τ , which is the isometry of the hyperbolic metrics involved.

Lemma 6.6 *For any $z \in (\Omega_- \cup \Omega_+) \setminus H^{-1}(\mathbb{R})$, we get that the hyperbolic metric expansion ratio*

$$\frac{|H^* d\rho_\infty|}{|d\rho_\infty|}(z) \geq \frac{|d\rho'|}{|d\rho_\infty|} = D_\iota^{-1}(z)$$

where ρ' is the hyperbolic metric of $\mathbb{C} \setminus H^{-1}(\mathbb{R})$ and ι is the inclusion map from $\mathbb{C} \setminus H^{-1}(\mathbb{R})$ into $\mathbb{C} \setminus \mathbb{R}$.

Proof. The second equality is obvious and the claim is equivalent to

$$\frac{|H^* d\rho_\infty|}{|d\rho'|}(z) \geq 1. \quad (21)$$

By Lemma 6.3, for every $k > 0$ we can represent both branches of H as compositions of H_{-k} defined on appropriate sets. For a composition of H_{-k} each inverse branch ζ can be defined from \mathbb{H}_+ or \mathbb{H}_- intersected with $D(0, \tau^k R)$ is univalent onto some set W_ζ which is contained in $\tau^k(\Omega_- \cup \Omega_+)$. Domains W_ζ cover the domain of H and ζ are analytic continuations of inverse branches of H . If ρ_W denotes the hyperbolic metric of W and ρ_k the hyperbolic metric of $D(0, \tau^k R)$, then $|d\rho_W| = |H^* d\rho_k|$. Since $W \subset (\mathbb{C} \setminus \mathbb{R}) \setminus H^{-1}(\mathbb{R})$, $|d\rho_W| \geq |d\rho'|$ and

$$\frac{|H^* d\rho_k|}{|d\rho'|}(z) \geq 1.$$

But when $k \rightarrow \infty$, then $|d\rho_k| \rightarrow |d\rho_\infty|$ from which estimate (21) follows.

□

We will write $D_{\rho_\infty} f = \frac{|f^* d\rho_\infty|}{|d\rho_\infty|}$.

Lemma 6.7 *Suppose that z_0 belongs to $(\Omega_- \cup \Omega_+) \setminus (\Omega_{c,-} \cup \Omega_{c,+})$, see Definition 4.1. Then for every $M > 0$ there exists $r > 0$ so that the image of the hyperbolic ball centered at z_0 with radius r with respect to the metric ρ_0 by the branch of H which is defined at z_0 contains the set $W_M = \{w \in \mathcal{D}^*(x_0, \tau x_0) : |\Re \log \frac{w}{H(z_0)}| < M\}$.*

Proof. To fix attention, the $z_0 \in \Omega_-$. For this proof, we will restrict H to Ω_- . Furthermore, from the definition of the \mathcal{EWF} class, H can be extended analytically as a map from some set $U \subset \mathcal{D}(x_0, \tau x_0)$ onto $D(0, R) \setminus \{0\}$ with $R > \tau x_0$.

Recall that $h = \log H$ can be well defined at the lifting of \exp to the universal cover of $\mathcal{D}^*(x_0, \tau x_0)$ by H . We will choose the lifting so that h is symmetric with respect to the real line. h is univalent and non-contracting from ρ_0 to the hyperbolic metric ρ' of \mathbb{H}_+ or \mathbb{H}_- intersected with $\{w : \Re w < \log R\}$. By the hypothesis of the Lemma, $|\Im h(z_0)| \geq \pi$. To fix attention, let $\Im h(z_0) = a > 0$. Then the diameter of the set $\log W_M \cap \{w : a - \frac{\pi}{2} \leq \Im w \leq a + \frac{3\pi}{2}\}$ with respect to ρ' is bounded depending only on M and r can be chosen equal to this bound. □

Uniform expansion. Now take any point $z \in \mathbb{C}$ which is never mapped to \mathbb{R} by the untempered tower dynamics. Proposition 7 then delivers a sequence z_n . Let χ_n be the corresponding tower iterate which maps z to z_n .

Lemma 6.8 *For every D there exists $\lambda > 1$, such that for every n and every w in the ball centered at z_n with radius D with respect to ρ_∞ , $D_{\rho_\infty} H_{-m_n}(w) > \lambda$, provided that w is in the domain of H_{-m_n} .*

Proof. By Lemma 6.6 and Fact 6.1, $DH_{\rho_\infty}(w) > \lambda(w) > 1$ where $\lambda(w)$ depends only on the distance in ρ_∞ from w to $H_{-m_n}^{-1}(\mathbb{R})$. But if either case of the alternative in Proposition 7 holds, then points z_n are all in a uniformly bounded ρ_∞ -distance from the corresponding set $H_{m_n}^{-1}(\mathbb{R})$. This is clearly true in the first case of the alternative. In the second case z_n we will apply Lemma 6.7 with the obvious rescaling by τ^{-m_n} , with $z_0 = z_{m_n}$ and any fixed positive M , since any ring will intersect the real line. We get a bounded hyperbolic distance from z_n to $H_{-m_n}^{-1}(\mathbb{R})$ and also from w to $H_{-m_n}^{-1}(\mathbb{R})$ by the triangle inequality. The claim follows from Lemma 6.6 and Fact 6.1. □

Lemma 6.9 *For every n , let ζ_n denote the inverse branch of χ_n which maps z_n to z defined on some simply-connected set $U_n \ni z_n$. Then for every D and ε there exists n_0 such that for every $n \geq n_0$ if the diameter of U_n with respect to ρ_∞ does not exceed D , then $\zeta_n(U_n)$ is inside the hyperbolic ball of radius ε centered at z .*

Proof. Pulling back a U_n will not increase its diameter, so each time we pass z_m its radius will be shrunk by a definite factor.

□

Proof of Theorem 7. Let $z_0 \in K_H$. If the orbit of z_0 by H ever enters the real line, then the claim of the Theorem follows from Proposition 3. If not, recall Lemma 4.7. If the distance from the orbit of z_0 to \mathbb{R} is positive, then quite obviously that $\rho_0(H^n(z_0), H^{-1}((x_0, \tau x_0)))$ is bounded independently of n . In the remaining case, by Lemma 4.7 we get a sequence m_n such that z_{m_n} is not in $\Omega_{-,c} \cup \Omega_{+,c}$. Then we use Lemma 6.7 with $z_0 := z_{m_n}$ and any M fixed and positive to get that $\rho_0(H^{m_n}(z_0), H^{-1}((x_0, \tau x_0)))$ is bounded independently on n . So, setting $m_n := n$ in the first case, we can now proceed in a uniform way.

By a reasoning used in the proof of Lemma 6.9, we get that the inverse branch H^{-m_n} which tracks back the orbit of z_0 shrinks the metric ρ_0 at a fixed rate. Thus, $\rho_0(z_0, H^{-m_n}((x_0, \tau x_0)))$ shrinks exponentially fast with n . Finally, preimages of $\tau^{-1}x_0$ are dense in $(x_0, \tau x_0)$ by Proposition 3.

Density of the Julia sets.

Proposition 8 *In the tower of the untempered complex dynamics H for every EWF-map, the Julia set*

$$\bigcup_{n=1}^{\infty} \tau^n K_H$$

is dense in \mathbb{C} .

We can now prove Proposition 8. For some fixed D and every n , we can find an element of $H_{-m_n}^{-1}(\mathbb{R})$, moreover, a preimage of 0 by H_{-m_n} , which can be joined to z_n by a simple arc γ_n of hyperbolic length which does not exceed some fixed D and which is completely contained in $\tau^{m_n}(\Omega_- \cup \Omega_+)$. This follows from Lemma 6.7 after rescaling. We can then find k which is at least equal to m_n and large enough so that the tower iterate χ_n can be represented as an iterate of H_{-k} .

Then the inverse branch ζ_n is defined on a neighborhood of γ_n . We can apply Lemma 6.9 to get that ζ_n maps γ_n into a neighborhood of z whose diameter shrinks to 0 as n grows. Letting n go to ∞ , we get that every ball centered z contains a preimage of 0 by some iterate of the tower dynamics. But every preimage of 0 in the tower belongs to some K_{H_k} and so Proposition 8 follows.

7 Rigidity of Towers

7.1 Conjugacy between towers.

Given the untempered dynamics and their towers built for two \mathcal{EWF} -maps, H and \hat{H} , we initially construct a quasiconformal conjugacy *between the towers*.

Proposition 9 *There is a quasi-conformal homeomorphism Υ of the plane, symmetric w.r.t. the real axis, and normalized so that $\Upsilon(0) = 0, \Upsilon(1) = 1, \Upsilon(\infty) = \infty$, which conjugates every H_{-n} with \hat{H}_{-n} : $\Upsilon \circ H_{-n} = \hat{H}_{-n} \circ \Upsilon$ whenever both sides are defined. Moreover, $\Upsilon(z) = \hat{\tau}\Upsilon(z/\tau)$ for any $z \in \mathbf{C}$.*

We initially construct the conjugacy between the *tempered* towers, H_{-n}^t, \hat{H}_{-n}^t . The conjugacy Υ^0 is between H_0^t and \hat{H}_0^t is obtained by Theorem 6. Denote $\Upsilon^n(z) = \hat{\tau}^n \Upsilon^0(\tau^{-n}z)$. For every n , we have

$$\Upsilon^n H_{-n}^t(z) = \hat{\tau}^n \Upsilon_0(\tau^{-n} \tau^n H^t(\tau^{-n}z)) = \hat{\tau}^n \hat{H}^t(\Upsilon_0(\tau^{-n}z)) = \hat{H}_{-n}^t(\Upsilon^n(z))$$

and so Υ^n conjugates H_{-n}^t to \hat{H}_{-n}^t . By Lemma 6.4, the same Υ^n also conjugates H_{-i}^t to \hat{H}_{-i}^t for $i = 0, \dots, n$.

Using the compactness of the family Υ^n , we pick a limit point Υ which conjugates the whole towers.

Two things need to be checked: that Υ also conjugates between *untempered* towers and that it is invariant under the rescaling.

Uniqueness of the conjugacy on the Julia set.

Lemma 7.1 *Suppose that H is a tempered dynamics built for some \mathcal{EWF} -map. Let Υ be a homeomorphism which self-conjugates H , i.e. $\Upsilon(H(z)) = H(\Upsilon(z))$ for every $z \in U_- \cup U_+$. In addition, Υ fixes 0, is symmetric about the real line and preserves its orientation. Then $\Upsilon(z) = z$ for every $z \in \tilde{K}_H$.*

Proof. Let $I_0 = (x_0, \tau x_0)$. \bar{I}_0 is the domain of circle dynamics generated by H . By Proposition 3, the orbit of 0 is dense in this interval, and so $H(z) = z$ for all $z \in I_0$. Now consider the set $\mathcal{I} = \cup_{k=0}^{\infty} H^{-k}(\bar{I}_0)$. We distinguish regular points mapped into I_0 by H^k for some $k \geq 0$. Regular points constitute a disjoint union of arcs, each of which is mapped onto I by some iterate of H . Branching points are preimages of x_0 and τx_0 and arcs of regular points join there.

Let z_0 be regular point and $z_1 = \Upsilon(z_0)$. If z_1 and z_0 belong to the same regular arc, then $z_1 = z_0$ since H^k is one-to-one on regular arcs and Υ is the identity of I_0 .

Point z_1 cannot be a branching point, since the conjugacy must permute branching points among themselves. If it belongs to a different regular arc, then

we construct an isotopy Υ_t between Υ and id by scaling the Beltrami coefficient of Υ by t . Each Υ_t conjugates H to some H_t which is a complex dynamics symmetric about \mathbb{R} . $\mathcal{I}_t = \Upsilon_t(\mathcal{I})$ is the full preimage of the domain of the circle dynamics for H_t and in particular, Υ_t still permutes among the branching points of \mathcal{I}_t . Then z_t has to join z_1 to z_0 inside $\Upsilon_t(\mathcal{I})$. Then, for some t , z_t has to pass through a branching point of $\Upsilon_t(\mathcal{I})$, but that is not possible by the foregoing remark.

Hence Υ fixes all regular points, but those are dense in \mathcal{I} . Finally, by Theorem 7, \mathcal{I} is dense in \tilde{K}_H .

□

Coming back to the proof of Proposition 9, we observe that for any $m > n$, $\Upsilon^n(z) = \Upsilon^m(z)$ provided that $z \in \tilde{K}_{H_{-n}}$. Indeed, both Υ^m and Υ^n conjugate H_{-n} to \hat{H}_{-n} and so $(\Upsilon^m)^{-1} \circ \Upsilon^n$ provides a self-conjugacy of H_{-n} and Lemma 7.1 becomes applicable.

Now if $\Upsilon = \lim_{k \rightarrow \infty} \Upsilon^{n_k}$, then $\Upsilon'(z) = \hat{\tau} \Upsilon(\tau^{-1}z)$ is the limit of the sequence Υ^{n_k+1} . For $z \in \tilde{K}_{H_{-m}}$ and any m , the values of both sequences at z stabilize. Hence, $\Upsilon(z) = \Upsilon'(z)$ for any $z \in \bigcup_{m=0}^{\infty} \tilde{K}_{H_{-m}}$ but this set is dense in \mathbb{C} by Proposition 8 and Lemma 6.2. So, $\Upsilon = \Upsilon'$ and invariance and the rescaling by τ has been demonstrated.

Finally, we need to show that Υ conjugates between the untempered towers as well. Given the invariance under τ , it will be enough to show that for any $z \in \Omega_- \cup \Omega_+$ we have

$$\Upsilon \circ H(z) = \hat{H} \circ \Upsilon(z) \quad (22)$$

for the untempered dynamics.

Assume first that $z \in \bigcup_{n=0}^{\infty} \tau^n K_H$. Then $z \in K_{H_{-n}}$ for all n large enough. But $K_{H_{-n}} = \tilde{K}_{H_{-n}}$ by Lemma 6.2 and so Υ conjugates on the entire forward orbit of z by H_{-n}^t , which is the same as the forward orbit by H_{-n} . But $H(z)$ is somewhere in this orbit by Lemma 6.4 and so relation (22) holds.

To finish the proof, invoke Proposition 8.

7.2 Invariant line-fields

We will identify *measurable line-fields* with differentials in the form $\nu(z) \frac{d\bar{z}}{dz}$ where ν is a measurable function with values on the unit circle or at the origin. A line-field is considered *holomorphic* at z_0 if for some holomorphic function ψ defined on a neighborhood of z_0 , we have $\nu(z) = c \frac{\psi'(z)}{\psi'(z)}$ for some constant c .

By a standard reasoning, Proposition 9 gives us a measurable line-field $\mu(z) \frac{d\bar{z}}{dz}$ which is invariant under the action of H_n^* for any n as well as under rescaling: $\mu(\tau z) = \mu(z)$.

We will proceed to show that μ must be trivial, i.e. 0 almost everywhere. This will be attained by a typical approach: showing first that μ cannot be non-trivial

and holomorphic at any z_0 for dynamical reasons, and on the contrary, that it must be holomorphic at some point for analytic reasons and because of expansion.

Absence of line-fields holomorphic on an open set.

Lemma 7.2 *The line-field μ cannot be both holomorphic and non-trivial on any open set.*

Proof. Let μ be holomorphic in a neighborhood W . Since μ is invariant under $z \mapsto z/\tau$ and since $\cup_{n \geq 0} \tau^n K_H$ is dense in the plane, one can assume that W is a neighborhood of a point b of K_H . Moreover, since b is approximated by preimages of x_0 and τx_0 , one can further assume that W is a neighborhood of a , such that $H^n(a) = x_0$ or $H^n(a) = \tau x_0$ for some $n \geq 0$, and (shrinking W) that H^n is univalent on W . To fix attention, suppose that $H^n(a) = x_0$, since the other case can be treated in the same way. Apply H^n and see that μ is holomorphic in a neighborhood W' of x_0 . Applying H one more time to $W' \cap U$, one sees that μ is holomorphic in a neighborhood of every point of a punctured disk $D(0, r) \setminus \{0\}$. Now apply the rescalings $z \mapsto \tau^n z$, $n = 0, 1, \dots$. Hence, μ is holomorphic everywhere except for 0. In particular, μ is holomorphic around $1 = H(0)$. Since H is univalent in a neighborhood of 0, then μ is actually holomorphic in the whole disc $D(0, r)$. Then μ cannot be holomorphic around $H^{-1}(0) = x_0$, a contradiction. □

Construction of holomorphic line-fields. Our goal is to prove the following:

Proposition 10 *Suppose that H is an untempered dynamics derived from the \mathcal{WF} -class. Assume that H fixes an invariant line-field $\mu(z) \frac{d\bar{z}}{dz}$, which is additionally invariant under rescaling: $\mu(\tau z) = \mu(z)$. Then the line-field is holomorphic at some point. Additionally, it is non-trivial in a neighborhood of the same point unless $\mu(z)$ vanishes almost everywhere.*

Construction of holomorphic line-fields is based on the following analytic idea. This lemma appeared in [12] and is repeated here for the sake of completeness.

Lemma 7.3 *Consider a line-field $\nu_0 \frac{d\bar{z}}{dz}$ defined on a neighborhood of some point z_0 which also is a Lebesgue (density) point for ν_0 . Consider a sequence of univalent functions ψ_n defined on some disk $D(z_1, \eta_1)$ chosen so that for every n and a fixed $\rho < 1$ the set $\psi_n(D(z_1, \rho\eta_1))$ covers z_0 . In addition, let $\lim_{n \rightarrow \infty} \psi'_n(z_1) = 0$. Define*

$$\mu_n(z) \frac{d\bar{z}}{dz} = \psi_n^*(\nu_0(w)) \frac{d\bar{w}}{dw}$$

Then for some subsequence n_k and a univalent mapping ψ defined on $D(z_1, \eta_1)$, $\mu_{n_k}(z)$ tend to $\nu_0(z_0) \frac{\overline{\psi'(z)}}{\psi'(z)}$ on a neighborhood of z_1 .

Proof. Let us normalize the objects by setting $\hat{\psi}_n := |\psi'_n(z_1)|^{-1} \psi_n$ and $\hat{\nu}_n(w) = \nu_0(|\psi'_n(z_1)|w)$. By bounded distortion, $\hat{\psi}_n(D(z_1, \rho\eta_1))$ contains some $D(z_0, r_1)$ and is contained in $D(z_0, r_2)$ with $0 < r_1 < r_2$ independent of n . By choosing a subsequence, and taking into account compactness of normalized univalent functions and the fact that z_0 was a Lebesgue point of ν_0 , we can assume that $\hat{\psi}_n$ converge to a univalent function ψ and $\hat{\nu}_n$ converge to a constant line-field $\nu_0(z_0) \frac{d\bar{w}}{dw}$ almost everywhere. Since

$$\mu_n(z) \frac{d\bar{z}}{dz} = \hat{\psi}_n^*(\hat{\nu}_n(w) \frac{d\bar{w}}{dw})$$

for all n , we get

$$\mu_n(z) \frac{d\bar{z}}{dz} \rightarrow \psi^*(\nu_0(z_0) \frac{d\bar{w}}{dw})$$

for $z \in D(z_1, \eta_1\rho)$ which concludes the proof of the Lemma. \square

Proof of Proposition 10. Start with a Lebesgue point z_0 of μ . If the field is non-trivial, without loss of generality $\mu(z_0) \neq 0$. Also, we can pick z_0 so that it is never mapped on the real line and we can use Proposition 7.

We then pick a sequence z_n from Proposition 7 in such a way that the same case happens for all n . In the first case, we choose a point Z to be an accumulation point of $\tau^{-m_n} z_n$. Without loss of generality, we suppose that $\tau^{-m_n} z_n \rightarrow Z$. The distance from Z to ω_∞ is positive and we can denote it by $2\eta_1$. Then, for any n we can find an inverse branch ζ_n of the tower iterate χ_n mapping z_0 to z_{m_n} defined on $D(z_{m_n}, \tau^{m_n}\eta)$. One easily checks that functions $\psi_n(z) = \zeta_n(\tau^{m_n} z)$ defined on $D(Z, \eta_1)$ satisfy the hypotheses of Lemma 7.3. In particular, their derivatives go to 0 because $D_{\rho_\infty} \chi_n(z_0)$ go to ∞ by Lemma 6.9.

To consider the second case of Proposition 7, fix attention on some n . The first observation is that without loss of generality $|H_{-m_n}(z_n)| < R'|\tau|^{m_n}$ with some $R' < R$ independent of n . Indeed, all points on the circle $C(0, |\tau|^{m_n} R)$ are in distance $\eta|\tau|^{m_n}$ from ω_∞ for some η positive. So if this additional property fails for infinitely many n , we can reduce the situation to the first case already considered.

Now the key observation is that for every n the point z_n has a simply connected neighborhood Y_n , a point $y_n \in Y_n$ such that the distance in the hyperbolic metric of Y_n from z_n to y_n is bounded independently of n . Finally, Y_n is mapped univalently by H_{-m_n} so that for some integer p_n and $\eta > 0$ which is independent of n its image covers $\tau^{p_n}(D(i, \eta))$ with $H_{-m_n}(y_n) = \tau^{p_n} i$. To choose Y_n invoke Lemma 6.7 after rescaling. In Lemma 6.7, set $M = \log |\tau|$ which will guarantee that W_M contains

$\tau^{-p_n}i$. Then Lemma 6.7 provides a uniform hyperbolic bound. Univalence can be obtained by restricting to the appropriately smaller neighborhood of the path joining z_{m_n} to $\tau^{p_n}i$.

Once y_n, Y_n, p_n were chosen, we easily conclude the proof. Let $R_n : D(0, 1) \rightarrow Y_n$ be Riemann maps of regions Y_n with $R_n(0) = y_n$. Then we can set $\psi_n = (\chi_n)^{-1} \circ R_n$ where χ_n are maps specified in Proposition 7. Maps ψ_n satisfy the conditions of Lemma 7.3. In particular, $|R_n^{-1}(z_n)|$ is bounded independently of n as a consequence of the construction of Y_n .

From this and Proposition 7, the derivatives of ψ_n at $R_n^{-1}(z_n)$ go to 0, and then the same can be said of $\psi'_n(0)$ by bounded distortion. So, by passing to a subsequence, we get that $R_n^*(\mu(z)\frac{dz}{dz})$ tend a.e. to a holomorphic line-field $\nu\frac{dw}{dw}$ on a neighborhood of 0.

To finish the proof, we ignore the fact that a subsequence has been chosen and consider mappings $T_n := \tau^{-p_n}H_{-m_n} \circ R_n$ defined on the unit disk. We have $T_n^*(\mu(z)\frac{dz}{dz}) = R_n^*(\mu(z)\frac{dz}{dz})$ for every n . Maps T_n are all univalent and have been normalized so that $T_n(0) = i$ and the image of $D(0, 1)$ under T_n contains $D(i, \eta)$ for a fixed $\eta > 0$, but avoids 0. Then T_n is a compact family of univalent maps and has a univalent limit T . Then it develops that μ in a neighborhood of i is the image under T of the holomorphic line-field ν from a neighborhood of 0, hence is holomorphic.

Proof of Theorem 3. Suppose that class \mathcal{EWF} contains two mappings, ϕ and $\hat{\phi}$. We construct their towers (H_{-n}) and (\hat{H}_{-n}) . By Proposition 9, the towers are quasiconformally conjugated which gives rise a line-field invariant under H_{-n}^* . If this line-field is non-trivial, then by Proposition 10 it is holomorphic on some open set, but then one reaches a contradiction with Lemma 7.2. If the line-field is trivial, then the quasiconformal conjugacy is affine. But it fixes 0 and 1, hence $\phi = \hat{\phi}$.

Proof of Theorem 1. By Facts 1.1 and 1.2 we can pick k_ℓ so that for every $n_\ell \geq k_\ell$ and every ℓ , the C^0 distance between $\mathcal{R}^{n_\ell}(\psi_\ell)$ and the fixed point H_ℓ tends to 0 as $\ell \rightarrow \infty$. Then, by Theorems 4 and 5, every subsequence of $\mathcal{R}^{n_\ell}(\psi_\ell)$ has a subsequence convergent to a \mathcal{EWF} map. But by Theorem 3, it means that the whole sequence converges to the unique member of the \mathcal{EWF} -class.

8 Proof of Theorem 2

The idea is to introduce the presentation function Π_ℓ for finite ℓ and to present the post-critical set of H_ℓ as a “non-escaping set” of the presentation function on some proper subinterval of its domain of definition. For the limit map and for its presentation function Π , it is done in Lemma 4.3. Then we pass to the limit as

$\ell \rightarrow \infty$ using Theorem 1. It is important to keep in mind that $X_\ell \rightarrow \frac{x_0}{\tau^2}$ when this limit is taken. Thus, points $\tau_\ell^2 X_\ell$ and x_ℓ merge in the limit to a parabolic fixed point of G .

Fix a finite ℓ . Recall that the point X_ℓ is defined as the unique solution of the equation $\phi_\ell(X_\ell) = \tau_\ell X_\ell$, see Lemma 2.1. Recall also that x_ℓ denotes the critical point of ϕ_ℓ . Note that $\tau_\ell^2 X_\ell < x_\ell < X_\ell$. The map H_ℓ consists of a pair of maps. It is defined as the map ϕ_ℓ on $(\tau_\ell^2 X_\ell, \tau_\ell X_\ell]$ and $\tau_\ell \circ \phi_\ell \circ \tau_\ell^{-1}$ on $(\tau_\ell X_\ell, \tau_\ell^3 X_\ell)$.

Define the presentation function Π_ℓ similar to Π as follows. Π_ℓ is defined on the interval $(\tau_\ell^2 X_\ell, \tau_\ell^3 X_\ell)$. For $\tau_\ell^2 X_\ell < x \leq \tau_\ell X_\ell$, $\Pi_\ell(x) = \tau_\ell x$. For $\tau_\ell X_\ell < x < \tau_\ell^3 X_\ell$, $\Pi_\ell(x) = \tau_\ell \circ \phi_\ell \circ \tau_\ell^{-1}(x)$.

Like in the limit case, it is post-critically finite since $\Pi_\ell(\tau_\ell x_\ell) = 0$ and 0 is a repelling fixed point of Π_ℓ . Also, the points $1, \tau_\ell^{-1}$ form a periodic orbit under Π_ℓ .

Denote by ω_ℓ the post-critical set of H_ℓ . It is the closure of the forward iterates $H_\ell^i(0)$, $i \geq 0$, of the critical value 0.

As it follows directly from the definition of Fibonacci combinatorics for covering maps, we have the following fact:

$$\omega_\ell \subset [x_\ell, \tau_\ell x_\ell]. \quad (23)$$

Note that $[x_\ell, \tau_\ell x_\ell]$ is a proper subset of the domain of definition of H_ℓ . Using the fact (23) and repeating word by word the proof of Lemma 4.4 we get the following.

Lemma 8.1 *A point $x \in (x_\ell, \tau x_\ell)$ is equal to $H_\ell^j(0)$, $j > 0$, if and only if $\Pi_\ell^k(x) = 1$ for some $k \geq 0$.*

Similar to the limit case, we consider the first return map of Π_ℓ to the open interval (x_ℓ, X_ℓ) . Denote this first return map by R_ℓ . Since Π_ℓ^2 maps (x_ℓ, X_ℓ) onto $(\tau_\ell^2 X_\ell, 0)$ and $\Pi_\ell^2 = \tau_\ell \circ \phi_\ell$ on (x_ℓ, X_ℓ) , the following statement follows easily.

Lemma 8.2 *Denote $\psi_n = \tau_\ell^{2n+1} \phi_\ell$, $n = 0, 1, \dots$. For every $n \geq 0$, there exists an interval $K_\ell^n \subset (x_\ell, X_\ell)$ such that $\psi_n(K_\ell^n) = (x_\ell, X_\ell)$, $n = 0, 1, \dots$. The intervals K_ℓ^n are pairwise disjoint. Domain of definition of the map R_ℓ is the union of K_ℓ^n , $n \geq 0$, so that R_ℓ on K_ℓ^n is equal to ψ_n . Each branch of R_ℓ has negative Schwarzian derivative and extends diffeomorphically to a fixed interval which contains $[x_\ell, X_\ell]$ in its interior.*

The next statement is crucial. Let $\tilde{\omega}_\ell = \omega_\ell \cap [x_\ell, X_\ell]$. Introduce also the non-escaping set of R_ℓ :

$$J(R_\ell) = \{x \in (x_\ell, X_\ell) : R_\ell^n(x) \in (x_\ell, X_\ell), n \geq 0\}.$$

Lemma 8.3 (a) A point $x \in (x_\ell, X_\ell)$ is equal to $H_\ell^j(0)$, $j > 0$, if and only if $R_\ell^k(x) = 1/\tau_\ell$ for some $k \geq 0$. (b) $\tilde{\omega}_\ell$ coincides with the closure of the non-escaping set $J(R_\ell)$.

Proof. (a) follows directly from Lemma 8.1 and from the fact that R_ℓ is the first return map. In turn, (b) follows from (a) and from the fact that the preimages of any point in $J(R_\ell)$ by all R_ℓ^n , $n \geq 0$, are dense in $J(R_\ell)$. Then we use that $1/\tau_\ell \in J(R_\ell)$ because $\Pi_\ell^2(1/\tau_\ell) = 1/\tau_\ell$.

□

As $\ell \rightarrow \infty$, then $x_\ell \rightarrow x_0$ and $X_\ell \rightarrow x_0/\tau^2$. Moreover, given $n \geq 0$, the interval K_ℓ^n tends K^n , so that the intervals K^n , $n \geq 0$, form a partition of $(x_0, x_0/\tau^2)$. We can elaborate Lemma 4.2 as follows. The first return map R of the limit presentation function Π into $(x_0, x_0/\tau^2)$ is defined on $\cup_{n \geq 0} K^n$, and $R|_{K^n} = \tau^{2n+1}\phi$. Clearly, the closure of the non-escaping set $J(R)$ of R is the interval $[x_0, x_0/\tau^2]$, which is just the intersection $\tilde{\omega}$ of the post-critical set $\omega = [x_0, \tau x_0]$ of H with the range of R . In particular, Hausdorff dimension $HD(\tilde{\omega})$ of $\tilde{\omega}$ is 1. On the other hand, since R is an infinite conformal iterated function system, see [15],

$$1 = HD(\tilde{\omega}) = \sup\{HD(J(R(A)))\},$$

where the supremum is taken over all finite subsets A of non-negative integers, $R(A)$ denotes the finite iterated function system formed by the subset of the branches of R with the indexes in the set A , and $J(R(A))$ is the closure of non-escaping set of $R(A)$. Given $\delta > 0$, we find A such that $HD(J(R(A))) > 1 - \delta$. On the other hand, by Theorem 1, any branch $\tau_\ell^{2n+1}\phi_\ell$ of R_ℓ tends to the corresponding branch $\tau^{2n+1}\phi$ of R . Therefore, since A is a finite set of indexes n , $HD(J(R_\ell(A))) \rightarrow HD(J(R(A)))$ as $\ell \rightarrow \infty$, with the obvious notation for $R_\ell(A)$ as a finite subset of R_ℓ with the indexes in the same finite set A . Thus, for any ℓ large enough,

$$HD(\omega_\ell) = HD(\tilde{\omega}_\ell) = HD(J(R_\ell)) \geq HD(J(R_\ell(A))) > 1 - \delta.$$

Since $\delta > 0$ is arbitrary, the theorem is proved.

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